February 6 Math 2306 sec. 51 Spring 2023

Section 5: First Order Equations Models and Applications

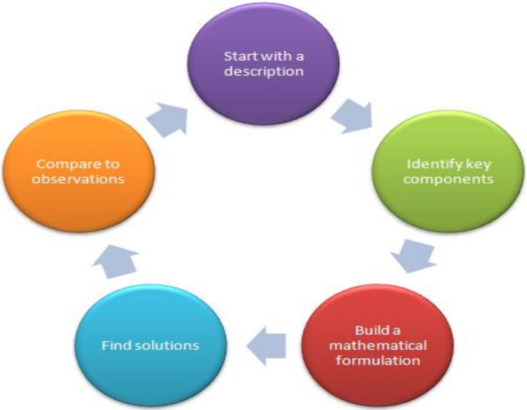

Figure: Mathematical Models give Rise to Differential Equations

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2021, there were 58 rabbits. In 2022, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2031.

We need variables. The population is changing in time, so let's introduce variables

$$t \sim \text{time} \quad \text{and} \quad P(t) \sim \text{is the population (density) at time } t.$$

We need to express the following mathematically:

The population's rate of change is proportional to the population.

$\frac{dP}{dt} \propto P$ i.e. $\frac{dP}{dt} = kP$ for constant k

Let's take t in years w/ $t=0$ in 2021

Then $P(0) = 58$ and $P(1) = 89$

We have $\frac{dP}{dt} = kP$, $P(0) = 58$ and $P(1) = 89$

Let's solve the IVP

$$\frac{dP}{dt} = kP, \quad P(0) = 58$$

Separate variables

$$\frac{1}{P}\frac{dP}{dt} = k$$

$$\frac{1}{P}\, dP = k\, dt$$

$$\int \frac{1}{P}\,dP = \int k\,dt = kt + C$$

Since $P > 0$

$$\ln P = kt + C$$

$$e^{\ln P} = e^{kt+C}$$

$$P = e^{kt} \cdot e^{C}$$

Let $A = e^{C}$ $\quad P(t) = Ae^{kt}$

Apply $P(0) = 58$ $\quad 58 = Ae^{0} \Rightarrow A = 58$

$$P(t) = 58\,e^{kt}$$

We can find k using $P(1) = 89$

$$89 = 58\, e^{k(1)} = 58\, e^{k}$$

$$\Rightarrow e^{k} = \frac{89}{58} \quad \Rightarrow \quad k = \ln\left(\frac{89}{58}\right)$$

$$\boxed{\text{The population is} \quad P(t) = 58\, e^{t \ln\left(\frac{89}{58}\right)}}$$

In 2031, $t=10$. This model predicts

$$P(10) = 58\, e^{10 \ln(89/58)} \approx 4198.055$$

The 2031 population would be about 4200

Exponential Growth or Decay

If a quantity P changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear. If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

Decay is usually written

$$\frac{dP}{dt} = -kP \qquad \text{w/} \quad k>0$$

Series Circuits: RC-circuit

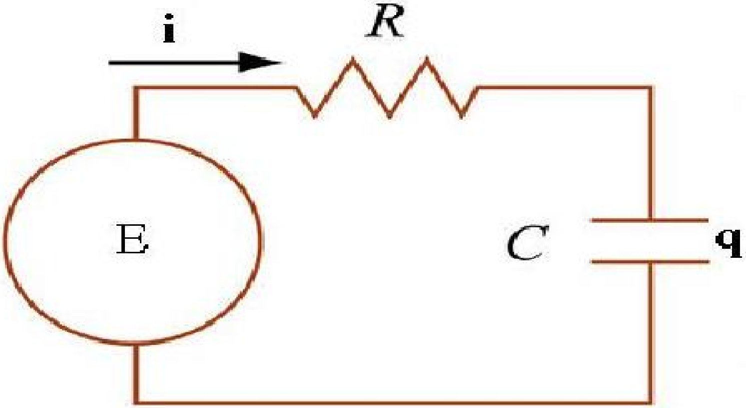

Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

Series Circuits: LR-circuit

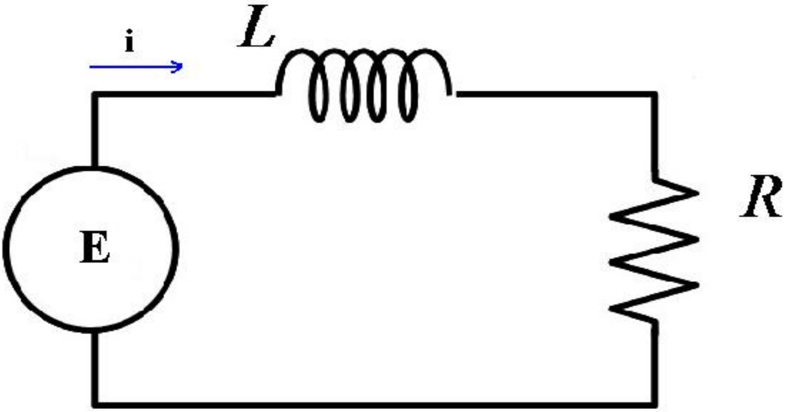

Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

Measurable Quantities:

Resistance R in ohms (Ω),	Implied voltage E in volts (V),
Inductance L in henries (h),	Charge q in coulombs (C),
Capacitance C in farads (f),	Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Kirchhoff's Law

The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force.

RC Series Circuit

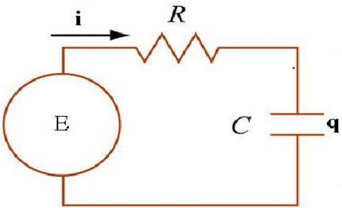

Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

drop across resistor	$+$	drop across capacitor	$=$	applied force
$R\frac{dq}{dt}$	$+$	$\frac{1}{C}q$	$=$	$E(t)$

$$\boxed{R\frac{dq}{dt} + \frac{1}{C}q = E(t)}$$

If $q(0) = q_0$, the IVP can be solved to find $q(t)$ for all $t > 0$.

LR Series Circuit

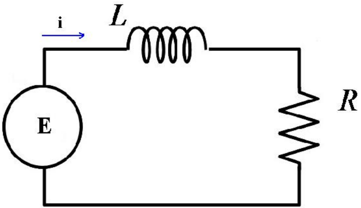

Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

$$\begin{array}{ccccc} \text{drop across inductor} & + & \text{drop across resistor} & = & \text{applied force} \\ L\frac{di}{dt} & + & Ri & = & E(t) \end{array}$$

$$\boxed{L\frac{di}{dt} + Ri = E(t)}$$

If $i(0) = i_0$, the IVP can be solved to find $i(t)$ for all $t > 0$.

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \to \infty$.

$$R\frac{dq}{dt} + \frac{1}{C}q = E$$

Here,

$E(t) = 200$

$R = 1000$

$C = 5 \cdot 10^{-6}$

The ODE is

$$1000\frac{dq}{dt} + \frac{1}{5 \cdot 10^{-6}}q = 200$$

$i(0) = q'(0) = 0.4$

$$\frac{\frac{1}{5 \cdot 10^{-6}}}{1000} = \frac{10^6}{5(10^3)} = \frac{10^3}{5} = 200$$

In standard form

$$\frac{dq}{dt} + 200q = \frac{1}{5}, \quad q'(0) = \frac{2}{5}$$

$$P(t) = 200, \quad \mu = e^{\int p(t)\,dt} = e^{\int 200\,dt} = e^{200t}$$

$$\frac{d}{dt}\left(e^{200t} q\right) = \frac{1}{5} e^{200t}$$

$$\int \frac{d}{dt}\left(e^{200t} q\right) dt = \int \frac{1}{5} e^{200t}\,dt$$

$$e^{200t} q = \frac{1}{5} \frac{1}{200} e^{200t} + k$$

$$q = \frac{\frac{1}{1000} e^{200t} + k}{e^{200t}}$$

$$q = \frac{1}{1000} + k e^{-200t}$$

Apply $q'(0) = \frac{2}{5}$, $q' = -200k e^{-200t}$

$$\frac{2}{5} = -200k e^{0} \Rightarrow k = \frac{2}{5} \cdot \frac{-1}{200} = \frac{-1}{500}$$

The charge on the capacitor

$$\boxed{q(t) = \frac{1}{1000} - \frac{1}{500} e^{-200t}}$$

The long time charge

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty} \frac{1}{1000} - \frac{1}{500} e^{-200t} = \frac{1}{1000}$$

$q \to \frac{1}{1000}$ Coulombs