February 7 Math 3260 sec. 52 Spring 2022

Section 1.5: Solution Sets of Linear Systems

- We defined a linear system, $A\mathbf{x} = \mathbf{b}$ as being **homogeneous** if the right hand side is the zero vector, i.e. $\mathbf{b} = \mathbf{0}$.
- A homogeneous system $A\mathbf{x} = \mathbf{0}$ always has at least one solution, $\mathbf{x} = \mathbf{0}$, called the **trivial solution.**
- And it will have nontrivial solutions if and only if it has at least one free variable.

Examples

We solved the system
$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

We set up and row reduced the augmented matrix to get

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We expressed the solution set (a line in $\mathbb{R}^3$) in **parametric vector form**

$$\mathbf{x} = s\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} \quad \text{where } s \text{ is any real number.}$$

Nonhomogeneous Systems

Find all solutions of the nonhomogeneous system of equations

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 7 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & -1 \\ 6x_1 & + & x_2 & - & 8x_3 & = & -4 \end{array}$$

We can use an augmented matrix

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equations for this rref are

$$x_1 - \frac{4}{3}x_3 = -1$$

$$x_2 = 2$$

The solution set is $\begin{aligned} x_1 &= -1 + \tfrac{4}{3}x_3 \\ x_2 &= 2 \\ x_3 &\text{ is free} \end{aligned}$

Let's write this in parametric vector form.

The solutions

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

We can write the solution $\vec{x}$ in parametric vector form as

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Solutions of Nonhomogeneous Systems

Note that the solution in this example has the form

$$\mathbf{x} = \mathbf{p} + t\mathbf{v}$$

with $\mathbf{p}$ and $\mathbf{v}$ fixed vectors and t a varying parameter. Also note that the $t\mathbf{v}$ part is the solution to the previous example with the right hand side all zeros. This is no coincidence!

$\mathbf{p}$ is called a **particular solution**, and $t\mathbf{v}$ is called a solution to the associated homogeneous equation.

Geometry $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ in $\mathbb{R}^3$

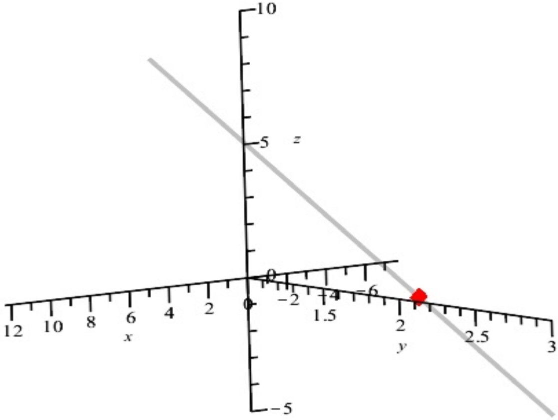

Figure: Plot of the line $\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$. The point $(-1, 2, 0)$ is shown in red.

Theorem

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for a given $\mathbf{b}$. Let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form

$$\mathbf{x} = \mathbf{p} + \mathbf{v}_h,$$

where $\mathbf{v}_h$ is any solution of the associated homogeneous equation $A\mathbf{x} = \mathbf{0}$.

We can use a row reduction technique to get all parts of the solution in one process.

Example

Find the solution set of the following system. Express the solution set in parametric vector form.

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & 2x_3 & + & 4x_4 & = & 1 \\ 2x_1 & + & 3x_2 & - & 6x_3 & + & 12x_4 & = & 4 \end{array}$$

We can use an augmented matrix.

$$\begin{bmatrix} 1 & 1 & -2 & 4 & 1 \\ 2 & 3 & -6 & 12 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -2 & 4 & 2 \end{bmatrix}$$

The system from the rref is

$$\begin{aligned} x_1 \qquad\qquad\qquad &= -1 \\ x_2 - 2x_3 + 4x_4 &= 2 \end{aligned}$$

The solutions are given by

$$x_1 = -1$$
$$x_2 = 2 + 2x_3 - 4x_4$$
x_3, x_4 are free

Going to parametric vector form.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2+2x_3-4x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -4x_4 \\ 0 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

we can express the solutions as

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

$$\vec{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

$$\vec{v}_h = s \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

Section 1.7: Linear Independence

We already know that a homogeneous equation $A\mathbf{x} = \mathbf{0}$ can be thought of as an equation in the column vectors of the matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_n\mathbf{a}_n = \mathbf{0}.$$

And, we know that at least one solution (the trivial one $x_1 = x_2 = \cdots = x_n = 0$) always exists.

Whether or not there is a nontrivial solution gives us a way to characterize the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Definition: Linear Independence

Definition: An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

If a set of vectors is not linearly independent, we say that it is **linearly dependent**.

Linear Dependence & Independence

We can restate this definition:

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists a set of weights $c_1, c_2, \ldots, c_p$, *at least one of which is nonzero*, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}.$$

Remark: The condition on the c's not all being zero is the same thing as saying the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}$ has a **nontrivial** solution.

Definition: An equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}$, with at least one $c_i \neq 0$, is called a **linear dependence relation**.

Theorem on Linear Independence

Theorem: The columns of a matrix A are linearly **independent** if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Remark: This follows directly from the definition of linear independence. It gives a characterization of the columns of a matrix as a set of vectors.

Example

Determine if the set is linearly dependent or linearly independent.

(a) $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

We can use the theorem above by creating a matrix $A = [\vec{v}_1\ \vec{v}_2]$. Now, we consider the homogeneous eqn. $A\vec{x} = \vec{0}$. Using an augmented matrix,

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & -2 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The system from the rref is $\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \end{aligned}$

There are no free variables, hence no nontrivial solutions.

The columns of A are linearly independent.

The set of vectors $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.

Example

Determine if the set is linearly dependent or linearly independent.

(b) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Notice that $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$. This can be arranged to give the <u>linear dependence relation</u>:

$$\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$$

This has the form

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

where $c_1 = c_2 = 1$ and $c_3 = \underline{-1}$.

Since at least one c is nonzero, we conclude that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is <u>linearly dependent</u>.