February 7 Math 3260 sec. 52 Spring 2024

Section 1.8: Intro to Linear Transformations

Definition

A transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

Definition

A transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, is **linear** provided

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v}$ in the domain of T, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every scalar c and vector $\mathbf{u}$ in the domain of T.

A Theorem About Linear Transformations:

Theorem:

If T is a linear transformation, then

(i) $T(\mathbf{0}) = \mathbf{0}$, and

(ii) $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$

for any scalars c, and d and vectors $\mathbf{u}$ and $\mathbf{v}$.

Remark: This second statement says:

> *The image of a linear combination is the linear combination of the images.*

It can be generalized to an arbitrary linear combination[1]

$$T(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) = c_1T(\mathbf{u}_1) + c_2T(\mathbf{u}_2) + \cdots + c_kT(\mathbf{u}_k).$$

[1] This is called the *principle of superposition*.

Comment on Notation

Recall that the vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ can be written using the notation

$$\mathbf{x} = (x_1, x_2, \dots, x_n).$$

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ might be written using this sort of notation. For example, if $T(\mathbf{x}) = \mathbf{y}$, this might be written like

$$T(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_m).$$

Example: For each transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

Determine

(i) The values of m and n, and
(ii) whether the transformation is linear or nonlinear.

(a) $T(x_1, x_2, x_3) = (x_1 - 2x_2, 1 - x_3)$ $\quad T: \mathbb{R}^3 \to \mathbb{R}^2$

$n = 3$, $\quad m = 2$

Check $T(\vec{0}) = T(0,0,0) = (0 - 2(0), 1 - 0)$

$= (0, 1) \neq \vec{0}$

T is not a linear transformation

Example

(b) $T(x_1, x_2) = (x_1 + 2x_2, 0, 0, 3x_2)$ $T : \mathbb{R}^2 \to \mathbb{R}^4$

$n = 2$ $m = 4$

To test linearity, check $T(\vec{0})$.

$T(0,0) = (0 + 2(0), 0, 0, 3(0)) = (0,0,0,0) = \vec{0}$

To check the properties, let

$\vec{u} = (a, b)$ $\vec{v} = (x, y)$ $a, b, x, y \in \mathbb{R}$.

$T(\vec{u}) = T(a,b) = (a + 2b, 0, 0, 3b)$

$T(\vec{v}) = T(x,y) = (x + 2y, 0, 0, 3y)$

$$\vec{u} + \vec{v} = (a+x, b+y)$$

$$T(\vec{u} + \vec{v}) = T(a+x, b+y) = (a+x+2(b+y), 0, 0, 3(b+y))$$

$$= (a+2b+x+2y, 0, 0, 3b+3y)$$

$$= (a+2b, 0, 0, 3b) + (x+2y, 0, 0, 3y)$$

$$= T(\vec{u}) + T(\vec{v})$$

Let k be a scalar, $k\vec{u} = (ka, kb)$

$$T(k\vec{u}) = T(ka, kb)$$

$$= (ka+2kb, 0, 0, 3kb)$$

$$= \left(k(a+2b),\ k(0),\ k(0),\ k(3b)\right)$$

$$= k\,(a+2b,\ 0,\ 0,\ 3b)$$

$$= k\,T(\vec{u})$$

Hence, T is a linear transformation.

An Example on $\mathbb{R}^2$

Let $r > 0$ be a scalar and consider the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = r\mathbf{x}.$$

This transformation is called a **dilation** if $r > 1$ and a **contraction** if $0 < r < 1$.

Exercise: Show that T is a linear transformation.

Let $\vec{u}, \vec{v}$ be in $\mathbb{R}^2$ and c be in $\mathbb{R}$.

$T(\vec{u}) = r\vec{u}$, $T(\vec{v}) = r\vec{v}$

$T(\vec{u} + \vec{v}) = r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$

$= T(\vec{u}) + T(\vec{v})$

$$T(c\vec{u}) = r(c\vec{u}) = c(r\vec{u})$$

$$= c\,T(\vec{u})$$

with those properties, T is a linear transformation.

The Geometry of Dilation/Contraction

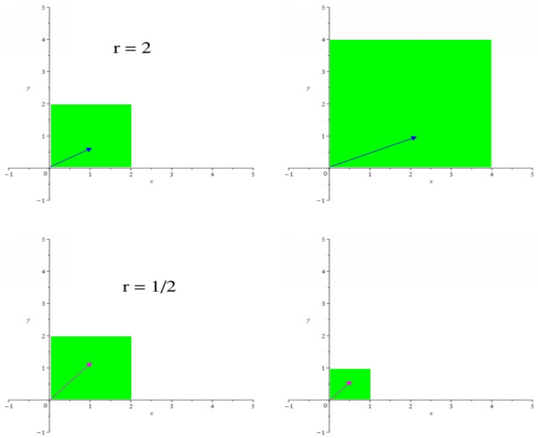

Figure: The 2×2 square in the plane under the dilation $\mathbf{x} \mapsto 2\mathbf{x}$ (top) and the contraction $\mathbf{x} \mapsto \frac{1}{2}\mathbf{x}$ (bottom). Each includes an example of a single vector and its image.

Example

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and for the vectors **u** and **v** in $\mathbb{R}^2$, it is known that

$$T(\mathbf{u}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{v}) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

Evaluate each of

1. $T(2\mathbf{u}) = 2T(\vec{u}) = 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$
2. $T\left(\frac{1}{4}\mathbf{v}\right) = \frac{1}{4}T(\vec{v}) = \frac{1}{4}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$
3. $T(3\mathbf{u} - 2\mathbf{v}) = 3T(\vec{u}) - 2T(\vec{v}) = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$