February 8 Math 2306 sec. 51 Spring 2023

Section 5: First Order Equations Models and Applications

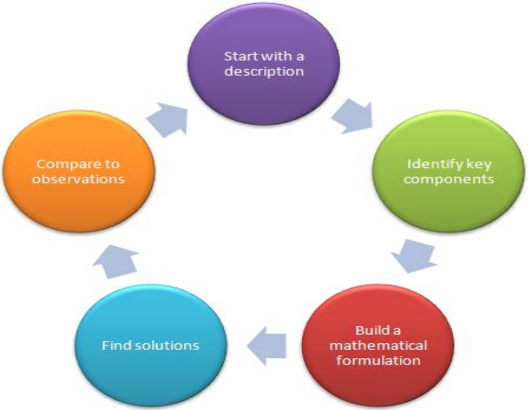

Figure: Mathematical Models give Rise to Differential Equations

Current Models

We have

Exponential Growth/Decay $\boxed{\dfrac{dP}{dt} = kP}$

RC-Series Circuit $\boxed{R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)}$

LR-Series Circuit $\boxed{L\dfrac{di}{dt} + Ri = E(t)}$

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

A Classic Mixing Problem

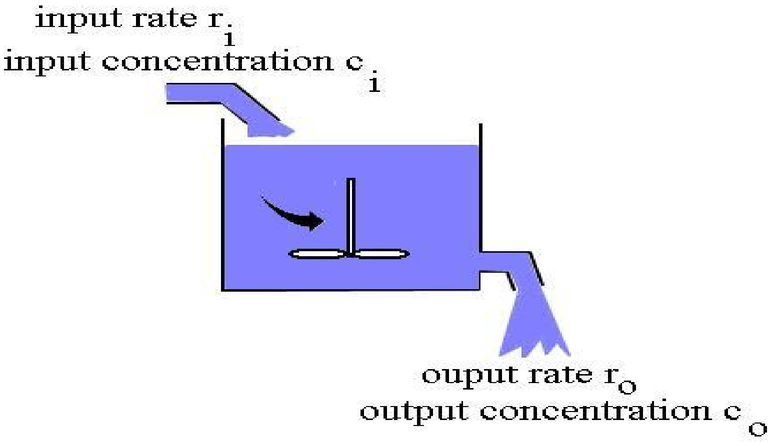

Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c} \textit{input rate} \\ \textit{of salt} \end{array} \right) - \left(\begin{array}{c} \textit{output rate} \\ \textit{of salt} \end{array} \right)$$

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i(c_i).$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o(c_o).$$

r_i, r_o, c_i are givens

Building an Equation

volume $V(t) = V(0) + r_i t - r_o t$

initial volume

The concentration of the outflowing fluid is

$$c_o = \frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

This equation is first order linear.

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

$V(0) = 500$ gal , $A(0) = 0$ lb

$c_i = 2 \frac{\text{lb}}{\text{gal}}$, $r_i = 5 \frac{\text{gal}}{\text{min}}$, $r_o = 5 \frac{\text{gal}}{\text{min}}$

$V(t) = V(0) + (r_i - r_o)t = 500 + (5-5)t = 500$ gal

$c_o = \frac{A}{V} = \frac{A}{500} \frac{\text{lb}}{\text{gal}}$

$$\frac{dA}{dt} = r_i c_i - r_o c_o$$

$$= 5(2) - 5 \cdot \frac{A}{500}$$

$$\frac{dA}{dt} + \frac{1}{100} A = 10 \quad , \quad A(0) = 0$$

1st order linear (and separable) IVP

$$P(t) = \frac{1}{100} \quad , \quad \mu = e^{\int p(t)\,dt} = e^{\int \frac{1}{100} dt}$$

$$= e^{\frac{1}{100} t}$$

$$\frac{d}{dt}\left(e^{\frac{1}{100} t} A \right) = 10\, e^{\frac{1}{100} t}$$

$$\int \frac{d}{dt}\left(e^{\frac{1}{100}t} A\right) dt = \int 10\, e^{\frac{1}{100}t}\, dt$$

$$e^{\frac{1}{100}t} A = 10(100)\, e^{\frac{1}{100}t} + k$$

$$A = \frac{1000\, e^{\frac{1}{100}t} + k}{e^{\frac{1}{100}t}}$$

$$A = 1000 + k\, e^{\frac{-1}{100}t}$$

Apply $A(0) = 0$

$$A(0) = 0 = 1000 + k e^{0}$$

$$\Rightarrow \quad k = -1000$$

The amount of salt is

$$A(t) = 1000 - 1000\, e^{-\frac{1}{100}t}$$

The concentration of salt in the tank after 5 minutes is

$$C = \frac{A(5)}{V(5)} \frac{\text{lb}}{\text{gal}} = \frac{1000 - 1000\, e^{-\frac{5}{100}}}{500} \approx 0.01 \ \frac{\text{lb}}{\text{gal}}$$

Let's look at the concentration as $t \to \infty$

$$\lim_{t\to\infty} \frac{A}{V} = \lim_{t\to\infty} \frac{1000 - 1000\, e^{-\frac{1}{100}t}}{500} = 2 \ \frac{\text{lb}}{\text{gal}}$$

$r_i \neq r_o$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

$r_i = 5$, $c_i = 2$, $r_o = 10$

The Volume

$$V(t) = V(0) + (r_i - r_o)t$$
$$= 500 + (5-10)t$$
$$= 500 - 5t \qquad \text{for } 0 \leq t \leq 100$$

$$c_o = \frac{A}{V}$$

$$\frac{dA}{dt} = r_i C_i - r_o C_o$$

$$= 5(2) - 10\left(\frac{A}{500 - 5t}\right)$$

$$= 10 - \frac{2A}{100 - t}$$

The new ODE is

$$\frac{dA}{dt} + \frac{2}{100-t} A = 10$$