February 8 Math 2306 sec. 52 Spring 2023

Section 5: First Order Equations Models and Applications

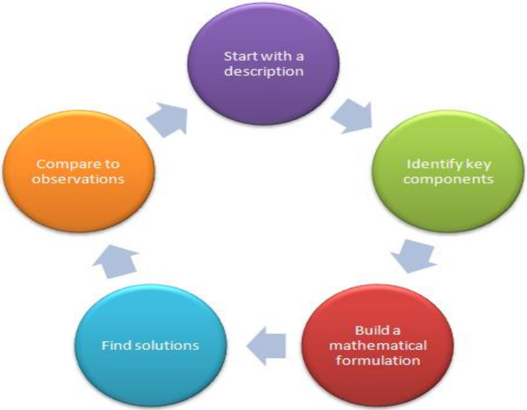

Figure: Mathematical Models give Rise to Differential Equations

Current Models

We have

Exponential Growth/Decay $$\boxed{\frac{dP}{dt} = kP}$$

RC-Series Circuit $$\boxed{R\frac{dq}{dt} + \frac{1}{C}q = E(t)}$$

LR-Series Circuit $$\boxed{L\frac{di}{dt} + Ri = E(t)}$$

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

A Classic Mixing Problem

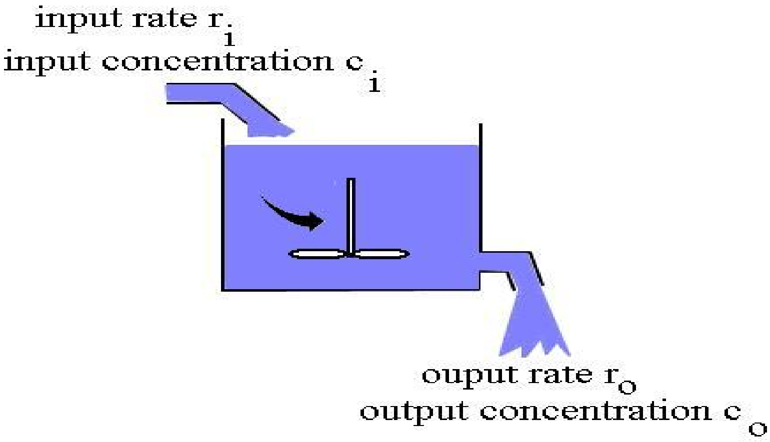

Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c} \text{input rate} \\ \text{of salt} \end{array}\right) - \left(\begin{array}{c} \text{output rate} \\ \text{of salt} \end{array}\right)$$

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i(c_i).$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o(c_o).$$

r_i, r_o, c_i are givens

Building an Equation

Volume $V(t) = V(0) + r_i t - r_o t$

initial volume ?.

The concentration of the outflowing fluid is

$$C_o = \frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

This equation is first order linear.

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i C_i \quad , A(0) = \text{initial salt}$$

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

$V(0) = 500$ gal , $A(0) = 0$ lb (pure water)

$c_i = 2 \frac{\text{lb}}{\text{gal}}$, $r_i = 5 \frac{\text{gal}}{\text{min}}$, $r_o = 5 \frac{\text{gal}}{\text{min}}$

The Volume: $V(t) = V(0) + (r_i - r_o)t$

$= 500 + (5-5)t = 500$ gal

$c_o = \frac{A}{V} = \frac{A}{500} \frac{\text{lb}}{\text{gal}}$

$$\frac{dA}{dt} = r_i c_i - r_o c_o$$

$$= 5(2) - 5\left(\frac{A}{500}\right)$$

$$\frac{dA}{dt} + \frac{1}{100} A = 10 \quad , \quad A(0) = 0$$

$$P(t) = \frac{1}{100} \quad , \quad \mu = e^{\int p(t)\,dt} = e^{\int \frac{1}{100}\,dt} = e^{\frac{1}{100} t} \, .$$

$$\frac{d}{dt}\left(e^{\frac{1}{100}t} A\right) = 10\, e^{\frac{1}{100}t}$$

$$\int \frac{d}{dt}\left(e^{\frac{1}{100}t} A\right) dt = \int 10\, e^{\frac{1}{100}t}\, dt$$

$$e^{\frac{1}{100}t} A = 10(100) e^{\frac{1}{100}t} + k$$

$$A = \frac{1000 e^{\frac{1}{100}t} + k}{e^{\frac{1}{100}t}}$$

$$A = 1000 + k e^{\frac{-1}{100}t}$$

general solution to the ODE

Apply $A(0) = 0$

$$A(0) = 0 = 1000 + k e^{0}$$

$$k = -1000$$

$$\boxed{\text{The amount of salt in the tank is} \quad A(t) = 1000 - 1000\, e^{-\frac{1}{100}t}}$$

After 5 minutes, the concentration in the tank is

$$C = \frac{A(5)}{V(5)} = \frac{1000 - 1000\, e^{-\frac{1}{100}(5)}}{500} \approx 0.01 \ \frac{\text{lb}}{\text{gal}}$$

Let's look at the concentration of salt in the tank as $t \to \infty$.

$$\lim_{t\to\infty} \frac{A}{V} = \lim_{t\to\infty} \frac{1000 - 1000e^{-\frac{1}{100}t}}{500} = 2 \quad \frac{\text{lb}}{\text{gal}}.$$

$r_i \neq r_o$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

$r_i = 5, \quad c_i = 2 \qquad r_o = 10$

$$V(t) = V(0) + (r_i - r_o)t$$
$$= 500 + (5-10)t$$
$$= 500 - 5t$$

$$c_o = \frac{A}{V} = \frac{A}{500 - 5t}$$

$$\frac{dA}{dt} = r_i C_i - r_o C_o$$

$$= 5(2) - 10\left(\frac{A}{500-5t}\right)$$

$$\frac{dA}{dt} + \frac{2}{100-t}A = 10$$