February 9 Math 3260 sec. 52 Spring 2022

Section 1.7: Linear Independence

Definition: An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. If a set of vectors is not linearly independent, we say that it is **linearly dependent**.

Definition: An equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$, with at least one $c_i \neq 0$, is called a **linear dependence relation**.

Theorem: The columns of a matrix A are linearly **independent** if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Example

Determine if the set of vectors is linearly dependent or independent. If dependent, find a linear dependence relation.

(c) $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$

$\vec{v}_1$ $\vec{v}_2$

We can create a matrix $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4]$ and consider the homogeneous equation $A\vec{x} = \vec{0}$.

The system has augmented matrix

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & 3 & 2 & 0 \\ 0 & 1 & 3 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

not a pivot column
x_4 is free

We have a free variable, hence $A\vec{x} = \vec{0}$ has non trivial solutions.

We conclude that its columns, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

To get a linear dependence relation, from the rref, we have

$$x_1 = -\frac{1}{3}x_4$$
$$x_2 = -2x_4$$
$$x_3 = \frac{2}{3}x_4$$
x_4 is free

Hence
$$-\frac{1}{3}x_4\vec{v}_1 - 2x_4\vec{v}_2 + \frac{2}{3}x_4\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$$

for any real number x_4. We get a linear dependence relation by choosing any nonzero value for x_4. Setting $x_4 = -3$, we get

$$\vec{V}_1 + 6\vec{V}_2 - 2\vec{V}_3 - 3\vec{V}_4 = \vec{0}$$

Ignore column 5 and set

$$[\vec{V}_1\ \vec{V}_2\ \vec{V}_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{V}_4$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{V}_4 = \frac{1}{3}\vec{V}_1 + 2\vec{V}_2 - \frac{2}{3}\vec{V}_3 \Rightarrow$$

$$\frac{1}{3}\vec{V}_1 + 2\vec{V}_2 - \frac{2}{3}\vec{V}_3 - \vec{V}_4 = \vec{0}$$

Theorem

Theorem: An indexed set of two or more vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the others in the set.

The last example is an example of this with $\vec{v}_4$ a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Example Problem

Let **u** and **v** be any nonzero vectors in $\mathbb{R}^3$. Show that if **w** is any vector in Span$\{\mathbf{u}, \mathbf{v}\}$, then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly **dependent.**

Note: This problem asks us to *prove* that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

We know that $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, but we don't know anything about what the entries are, and we know that **w** is in Span$\{\mathbf{u}, \mathbf{v}\}$.

Since $\vec{w}$ is in Span$\{\vec{u}, \vec{v}\}$, we know that

$\vec{w} = c_1\vec{u} + c_2\vec{v}$ for some numbers c_1 and c_2.

If we subtract $\vec{w}$ from both sides, we get a linear dependence relation

$$c_1\vec{u} + c_2\vec{v} - \vec{w} = \vec{0}.$$

Note the coefficient of $\vec{w}$ is -1 which is nonzero.

Hence $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent.

Caveat!

A set may be linearly dependent even if all proper subsets are linearly independent. For example, consider

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Each set $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. (You can easily verify this.)

However,

$$\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1 \quad \text{i.e.} \quad \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0},$$

so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

This means that you can't just consider two vectors at a time.

Two More Theorems

Theorem: If a set contains more vectors than there are entries in each vector, then the set is linearly **dependent**. That is, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a set of vector in $\mathbb{R}^n$, and $p > n$, then the set is linearly dependent.

For example, if you have 7 vectors,

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\},$$

and each of these is a vector in $\mathbb{R}^5$,

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \\ v_{41} \\ v_{51} \end{bmatrix} \quad \text{and so forth,}$$

then they must be **linearly dependent** because $7 > 5$.

Two More Theorems

Theorem: Any set of vectors that contains the zero vector is linearly **dependent.**

This says that no matter what the other vectors are, if the zero vector $\mathbf{0}$ is part of your set of vectors, the set is automatically **linearly dependent**.

Determine if the set is linearly dependent or linearly independent

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$

This set is dependent. There are 4 vectors from $\mathbb{R}^3$. $4 > 3$.

Determine if the set is linearly dependent or linearly independent

(b) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -8 \\ 1 \end{bmatrix}, \right\}$

$\vec{0}$ is included, this set is linearly dependent.