MATH 3260 sec. 56 January 8, 2025

Section 1.1: Overview

Recall that we are interested in working with systems of linear equation such as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & \vdots & \cdots & \vdots & & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array} \tag{1}$$

We immediately associate a pair of matrices with equation (1), the coefficient

and augmented matrices:

$$\text{Coefficient} \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{Augmented} \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Consistency

We defined a **solution** and **solution set** for a system of linear equations, and we saw that a linear system has one of the following:

i no solution, or

ii exactly one solution, or

iii infinitely many solutions.

We can a system **consistent** if it has at least one solution and **inconsistent** otherwise.

We said that two systems are **equivalent** if they have the same solution set.

Elementary Row Operations & Row Equivalence

Elementary Row Operations

We'll use the notation R_i to refer to the i^{th} row of a matrix. We have three row operations that we call **elementary row operations**.

i. Interchange row i and row j (**swap**), $R_i \leftrightarrow R_j$.
ii. Multiply row i by any nonzero constant k (**scale**), $kR_i \to R_i$.
iii. Replace row j with the sum of itself and k times row i (**replace**), $kR_i + R_j \to R_j$.

Definition: Row Equivalence

Two matrices are called **row equivalent** if one can be obtained from the other by performing a sequence of elementary row operations.

Theorem on Row Equivalence

Systems with Row Equivalent Augmented matrices are Equivalent

Theorem: If the augmented matrices of two linear systems of equations are row equivalent, then the linear systems of equations are equivalent (i.e., they have the same solution set).

This is a big deal! We exploit this to use matrices as a tool for solving linear systems. The structure of some matrices will give us a bunch of information—about linear systems as well as some other interesting stuff to come.

Example

For each augmented matrix, determine if the associated system is consistent or inconsistent. If consistent, identify the solution set.

(a) $$\left[\begin{array}{cccc} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Example

For each augmented matrix, determine if the associated system is consistent or inconsistent. If consistent, identify the solution set.

(b) $\left[\begin{array}{cccc} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 3 \end{array}\right]$

Example

For each augmented matrix, determine if the associated system is consistent or inconsistent. If consistent, identify the solution set.

(c) $\begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Section 1.2: Row Reduction and Echelon Forms

Definition: Echelon Form

A matrix is in **echelon form**, also called *row echelon form (ref)*, if it has the following properties:

i Any row of all zeros are at the bottom.

ii The first nonzero number (called the *leading entry*) in a row is to the right of the first nonzero number in all rows above it.

iii All entries below a leading entry are zeros.[a]

[a] This condition is superfluous but is included for clarity.

an ref

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$$

not an ref

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Reduced Echelon Form

Definition: Reduced Echelon Form

A matrix is in **reduced echelon form**, also called *reduced row echelon form (rref)* if it is in echelon form and has the additional properties

iv The leading entry of each row is 1 (called a *leading* 1), and

v each leading 1 is the only nonzero entry in its column.

an rref

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

not an rref

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Some Remarks

- Given any matrix—augmented or otherwise, we can perform row ops to obtain an echelon form (or reduced echelon form).
- A matrix can be row equivalent to various echelon forms.
- In practice, we usually aim for a reduced echelon form—i.e., we don't stop doing operations until we hit an rref. (There may be exceptions depending on our goal.)
- We'll generally take an organized, methodical approach to row reduction, but it involves making choices.

Pivot Positions & Pivot Columns

Uniqueness of RREF

Theorem: A given matrix is row equivalent to exactly one reduced echelon form.

That is, a given matrix is row equivalent to many different refs but to only ONE rref! This allows for the following unambiguous definitions.

Pivots

Definition: A **pivot position** in a matrix A is a location that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

Identifying Pivot Positions and Columns

The following matrices are **row equivalent**. Identify the pivot positions and pivot columns of the matrix A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Complete Row Reduction isn't needed to find Pivots

The following three matrices are row equivalent. (Note, B is an ref but not an rref, and C is an rref.)

$$A=\begin{bmatrix} 1 & 1 & 4\\ -2 & 1 & -2\\ 1 & 0 & 2\end{bmatrix},\quad B=\begin{bmatrix} 1 & 1 & 4\\ 0 & 3 & 6\\ 0 & 0 & 0\end{bmatrix}\quad \text{and}\quad C=\begin{bmatrix} 1 & 0 & 2\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix}$$

Identify the pivot positions and pivot columns of the matrix A.

Row Reduction Algorithm

We'll follow a set of steps to obtain an rref. There are two main phases:

- **Forward:** We work from left to right, top down to obtain an ref.
- **Backward:** From an ref, we work from right to left, bottom up to clear nonzero entries above the pivot positions.

Example: We will row reduce the following matrix.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Step 1: The left most nonzero column is a pivot column. Its top position is a pivot position

Row Reduction Algorithm

Step 2: Select a nonzero entry in the pivot column as the pivot and use row swap (if needed) to move this entry to the top.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Row Reduction Algorithm

Step 3: Use row replacement operations to get zeros in all entries below the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Row Reduction Algorithm

Step 4: Ignore the row with a pivot, all rows above it, the pivot column, and all columns to its left, and repeat steps 1-3 on the resulting sub-matrix.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Row Reduction Algorithm

Step 5: From an ref, starting with the right most pivot and working up and to the left, use row operations to get a zero in each position above a pivot. Scale to make each pivot a 1.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Row Reduction Algorithm

Row Reduction Algorithm

Echelon Form & Solving a System

Recall: Row equivalent matrices correspond to equivalent systems.

Suppose the matrix on the left is the augmented matrix for a linear system of equations in the variables x_1, x_2, x_3, x_4, and x_5. Use the rref to characterize the solution set to the linear system.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Definition: Basic & Free Variables

Suppose a system has m equations and n variables, $x_1, x_2, \ldots, x_n$. The first n columns of the augmented matrix correspond to the n variables. For $1 \leq i \leq n$,

- If the i^{th} column is a pivot column, then x_i is called a **basic variable.**
- If the i^{th} column is NOT a pivot column, then x_i is called a **free variable.**

Basic & Free Variables

Consider the system of equations along with its augmented matrix.

$$\begin{array}{rcrcrcrcrcr} & & 3x_2 & - & 6x_3 & + & 6x_4 & + & 4x_5 & = & -5 \\ 3x_1 & - & 7x_2 & + & 8x_3 & - & 5x_4 & + & 8x_5 & = & 9 \\ 3x_1 & - & 9x_2 & + & 12x_3 & - & 9x_4 & + & 6x_5 & = & 15 \end{array}$$

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

We determined that the matrix was row equivalent to the rref

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Hence the **basic** variables are x_1, x_2, and x_5, and the **free** variables are x_3 and x_4.

Expressing Solutions

To avoid confusion, i.e., in the interest of clarity, we will **always** write solution sets by expressing basic variables in terms of free variables. We will not write free variables in terms of basic. That is, the solution set to the system whose augmented matrix is row equivalent to

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

will be written

$$\begin{aligned} x_1 &= 3 + 2x_3 \\ x_2 &= 2 + 2x_3 \\ x_3 & \quad \text{is free} \\ x_4 &= 0 \end{aligned}$$

This is called a *parametric* form or description of the solution set.

Proper Solution Set Expressions

We will never express free variables in terms of basic variables. All three of the following result from the same augmented matrix:

$$\begin{array}{lcl} x_1 & = & 3+2x_3 \\ x_2 & = & 2+2x_3 \\ x_3 & & \text{is free} \\ x_4 & = & 0 \end{array} \qquad \begin{array}{lcl} x_3 & = & -3/2+1/2x_1 \\ x_2 & = & 2+2x_3 \\ x_3 & & \text{is free} \\ x_4 & = & 0 \end{array} \qquad \begin{array}{lcl} x_1 & = & 3+2x_3 \\ x_2 & = & -1+x_1 \\ x_3 & & \text{is free} \\ x_4 & = & 0 \end{array}$$

The left most parametric description is correct. The two expressions in red are **not correct** descriptions. They both include convoluted descriptions of the relationships between the variables.

Consistent versus Inconsistent Systems

Consider each rref. Determine if the underlying system (the one with this as its augmented matrix) is consistent or inconsistent.

(a) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

An Existence and Uniqueness Theorem

Theorem A linear system is consistent if and only if the right most column of the augmented matrix is **NOT** a pivot column. That is, if and only if each echelon form **DOES NOT** have a row of the form

$$[0\ 0\ \cdots\ 0\ b], \quad \text{for some nonzero } b.$$

Moreover, if a linear system is consistent, then it has

(i) exactly one solution if there are no free variables, and
(ii) infinitely many solutions if there is at least one free variable.

Remark: If the **coefficient** matrix has non-pivot columns, the system is either inconsistent or has infinitely many solutions.