January 10 Math 3260 sec. 52 Spring 2022

A Random Motivational Example

In a certain city, ABC shipping has one receiving (A) and two distribution hubs (B & C). On a given day, 80 packages enter center A and will be distributed to hubs B and C for delivery. Twenty packages will go to a major client from hub C, the rest are to be distributed in quantities $x_1, \dots, x_4$ among the hubs and out for delivery.

Motivating Example

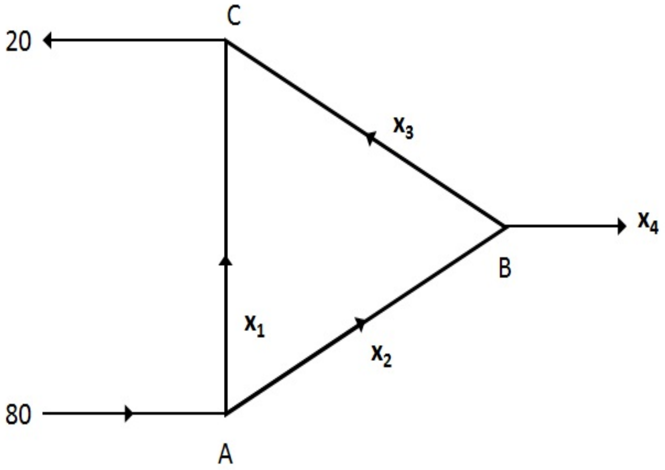

Figure: Distribution Scheme

Equations for Package Quantities

Assuming all of the packages are delivered to customers outside of the shipping company, the quantities $x_1, \ldots, x_4$ have to satisfy the equations

$$\begin{array}{rcrcrcrcr} x_1 & & & + & x_3 & & & = & 20 \\ & & x_2 & - & x_3 & - & x_4 & = & 0 \\ x_1 & + & x_2 & & & & & = & 80 \end{array}$$

Questions

- Is there a set of numbers $x_1, \ldots, x_4$ that satisfy all of the equations?
- If there is a set of numbers, is it the only one?
- If we could find numbers $x_1, \ldots, x_4$, and then the input 80 changed (say on another day), do we have to do all the work again? Or is there a way to generalize our finding?

(This is just to illustrate the kinds of questions addressed by **Linear Algebra.** We'll leave answering these questions for another day.)

We'll work in a variety of settings...

Linear sys.

$$\begin{array}{rcrcrcrcr} x_1 & & & + & x_3 & & & = & 20 \\ & & x_2 & - & x_3 & - & x_4 & = & 0 \\ x_1 & + & x_2 & & & & & = & 80 \end{array}$$

Matrix eqns.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ 80 \end{bmatrix}$$

More Matrices

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 80 \end{bmatrix}$$

Vector eqns.

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ 80 \end{bmatrix}$$

Two main abstractions we'll be interested in are **Linear Transformations** and **Vector Spaces**.

Section 1.1: Systems of Linear Equations

We begin with a linear (*algebraic*) equation in n variables x_1, x_2, ..., x_n for some positive integer n.

A **linear equation** can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

The numbers $a_1, \ldots, a_n$ are called the *coefficients*. These numbers and the right side b are real (or complex) constants that are **known**.

Linear Equation in n Variables

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

Notice the main structure on the left side. The unknowns/variables $(x_1, \ldots, x_n)$ are

- multipled by numbers (a.k.a. coefficients), and
- added together.

Other types of actions (squaring, multiplying variables, taking variable's reciprocal, etc.) aren't allowed if an equation is **linear**.

Examples of Equations that are or are not Linear

$$2x_1 = 4x_2 - 3x_3 + 5 \quad \text{and} \quad 12 - \sqrt{3}(x + y) = 0$$

These equations are linear.

The first can be written as

$$2x_1 - 4x_2 + 3x_3 = 5$$

the second is

$$\sqrt{3}\,x + \sqrt{3}\,y = 12$$

Examples of Equations that are or are not Linear

$$x_1 + 3x_3 = \frac{1}{x_2} \quad \text{and} \quad xyz = \sqrt{w}$$

These equations are NOT linear.

$\frac{1}{x_2}$ is a nonlinear term

xyz and $\sqrt{w}$ are nonlinear terms.

A *Linear System* is a collection of linear equations in the same variables

Example 1:

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & - & 3x_3 & + & x_4 & = & -3 \\ -x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & = & 8 \end{array}$$

Example 1 is a linear system that has two equations in four variables.

Example 2:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4 \\ 3x & + & 0y & + & 12z & = & 0 \\ 2x & + & 2y & - & 5z & = & -6 \end{array}$$

Example 2 is a linear system that has three equations in three variables.