January 11 Math 2306 sec. 52 Spring 2023

Section 1: Concepts and Terminology

Suppose $y = \phi(x)$ is a differentiable function. We know that $dy/dx = \phi'(x)$ is another (related) function.

For example, if $y = \cos(2x)$, then y is differentiable on $(-\infty, \infty)$. In fact,

$$\frac{dy}{dx} = -2\sin(2x).$$

Even dy/dx is differentiable with $\dfrac{d^2y}{dx^2} = -4\cos(2x)$.

Suppose $y = \cos(2x)$

Note that $$\frac{d^2y}{dx^2} + 4y = 0.$$

If $y = \cos(2x)$, then $y'' = -4\cos(2x)$

So $y'' + 4y =$

$-4\cos(2x) + 4\cos(2x) = 0$

$0 = 0$

That's a true statement, zero does equal zero.

A *differential* equation

The equation $\quad \dfrac{d^2y}{dx^2} + 4y = 0$

is an example of a **differential equation.**

Questions: If we only started with the equation, how could we determine that $\cos(2x)$ satisfies it?

Also, is $\cos(2x)$ the only possible function that y could be?

Definition

A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

Solving a differential equation refers to determining the dependent variable(s)—as function(s).

Independent Variable: will appear as one that derivatives are taken with respect to.

Dependent Variable: will appear as one that derivatives are taken of.

In $y = f(x)$, the y is dependent and the x is independent.

Classifications (ODE versus PDE)

Type: An **ordinary differential equation (ODE)** has exactly one independent variable[1]. For example

$$\frac{dy}{dx} - y^2 = 3x, \quad \text{or} \quad \frac{dy}{dt} + 2\frac{dx}{dt} = t,$$

$$\text{or} \quad y'' + 4y = 0$$

[1] These are the subject of this course.

Classifications (ODE versus PDE)

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

The expression $\dfrac{\partial y}{\partial t}$ is read

the partial derivative of y with respect to t.

It's computed by taking the derivative of $y = f(x,t)$ while keeping the other variable, x, fixed.

Classifications (Order)

Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

$$\frac{dy}{dx} - y^2 = 3x$$

1st order ODE

$$y''' + (y')^4 = x^3$$

3rd order ODE

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

2nd order PDE

Notations and Symbols

We'll use standard derivative notations:

$$\text{Leibniz:} \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \ldots \quad \frac{d^ny}{dx^n}, \quad \text{or}$$

$$\text{Prime \& superscripts:} \quad y', \quad y'', \quad \ldots \quad y^{(n)}.$$

Newton's **dot notation** may be used if the independent variable is time. For example if s is a position function, then

$$\text{velocity is } \frac{ds}{dt} = \dot{s}, \quad \text{and acceleration is } \frac{d^2s}{dt^2} = \ddot{s}$$

Notations and Symbols

An n^{th} order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

where F is some real valued function of $n + 2$ variables.

Our equation $\dfrac{d^2y}{dx^2} + 4y = 0$ has this form where

$$F(x, y, y', y'') = \frac{d^2y}{dx^2} + 4y.$$

Notations and Symbols

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}).$$

Our equation $\dfrac{d^2 y}{dx^2} + 4y = 0$ can be written in normal form.

$$\frac{d^2 y}{dx^2} = -4y \quad \text{note that} \quad f(x, y, y') = -4y$$

Classifications (Linear Equations)

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Example First Order:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Example Second Order:

$$a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Properties of a Linear ODE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

- Each of the coefficients $a_0, \ldots, a_n$ and the right hand side g may depend on the independent variable but not the dependent one.
- y, and its derivatives can only appear as themselves (not squared, square rooted, inside some other function).
- The characteristic structure of the left side is that

  $$y, \quad \frac{dy}{dx}, \quad \frac{d^2 y}{dx^2}, \ldots, \frac{d^n y}{dx^n}$$

  are multiplied by functions of the independent variable and added together.

Examples of Linear ODEs

$$y'' + 4y = 0 \qquad\qquad t^2\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - x = e^t$$

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$a_2(x) = 1$ $\qquad$ $a_2(t) = t^2$

$a_1(x) = 0$ $\qquad$ $a_1(t) = 2t$

$a_0(x) = 4$ $\qquad$ $a_0(t) = -1$

$g(x) = 0$ $\qquad$ $g(t) = e^t$

Examples of Nonlinear ODEs

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = x^3 \qquad\qquad u'' + u' = \cos u$$

$\left(\frac{dy}{dx}\right)^4$ is a nonlinear term

Cos u is a nonlinear term because u is dependent

Example: Classification

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

(a) $y'' + 2ty' = \cos t + y - y'''$

$\Rightarrow \quad y''' + y'' + 2ty' - y = \cos t$

It's linear

Order: 3rd

dep var: y

ind. var: t

(b) $\ddot{\theta}+\frac{g}{\ell}\sin\theta = 0$ g and ℓ are constant

Order: 2^{nd} order

dep. var: θ

Ind var: t, time

$\sin\theta$ is a nonlinear term

it's nonlinear

Solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ (*)

Solution or Explicit Solution

Definition: A function ϕ defined on an interval[2] I and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Example: $\phi(x) = \cos(2x)$ is a **solution** of $y'' + 4y = 0$ on $(-\infty, \infty)$ because

- it is twice differentiable, and
- when we set $y = \cos(2x)$ in the equation, it gives a true statement (namely, $0 = 0$).

[2] The interval is called the *domain of the solution* or the *interval of definition*.

Examples:

Verify that the given function is an solution of the ODE on the indicated interval. The c_1 and c_2 are constants.

$$\phi(x) = c_1 x + \frac{c_2}{x}, \quad I = (0, \infty), \quad x^2 y'' + xy' - y = 0$$

ϕ is twice diff'able on I

Set $y = \phi(x)$

$y = c_1 x + c_2 x^{-1}$

$y' = c_1 - c_2 x^{-2}$

$y'' = 2 c_2 x^{-3}$

$$x^2y'' + xy' - y = 0$$

$$x^2\left(2c_2x^{-3}\right) + x\left(c_1 - c_2x^{-2}\right) - \left(c_1x + c_2x^{-1}\right) \stackrel{?}{=} 0$$

$$2c_2x^{-1} + c_1x - c_2x^{-1} - c_1x - c_2x^{-1} \stackrel{?}{=} 0$$

Collect x and x^{-1}

$$x^{-1}\left(2c_2 - c_2 - c_2\right) + x\left(c_1 - c_1\right) \stackrel{?}{=} 0$$

$$0 = 0$$

So ϕ was twice differentiable and when we set y = ϕ and sub into the equation we get a true statement, zero = zero. This shows that ϕ is a **solution**.