January 12 Math 3260 sec. 51 Spring 2022

Section 1.1: Systems of Linear Equations

We defined a **linear (algebraic) equation** in n variables

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

and we defined a **linear system**

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
 & \vdots & & & \vdots & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m.
\end{array}$$

Some Preliminary Terms

- A **solution** is a list of numbers $(s_1, s_2, \ldots, s_n)$ that reduce each equation in the system to a true statement upon substitution.
- A **solutions set** is the set of all possible solutions of a linear system.
- Two systems are called **equivalent** if they have the same solution set.

Note: If the variables are $x_1, x_2, \ldots, x_n$, then a list such as $(s_1, s_2, \ldots, s_n)$ given as an n-tuple is understood to mean that upon substitution we set $x_1 = s_1$, $x_2 = s_2$, and so forth. If the variables are x, y, z, then (s_1, s_2, s_3) would mean $x = s_1$, $y = s_2$, and $z = s_3$.

An Example

$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & -1 \\ -4x_1 & + & 2x_2 & = & 2 \end{array}$$

(a) Show that $(1, 3)$ is a solution.

$(x_1, x_2) = (1, 3)$

In the 1st eqn, set $x_1 = 1$, $x_2 = 3$

$2 \cdot 1 - 3 = -\underline{1}$ true

and the 2nd eqn.

$-4 \cdot 1 + 2 \cdot 3 = 2$ true

An Example Continued

$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & -1 \\ -4x_1 & + & 2x_2 & = & 2 \end{array}$$

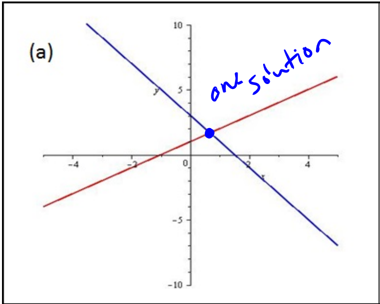
The **solution set** for this system is

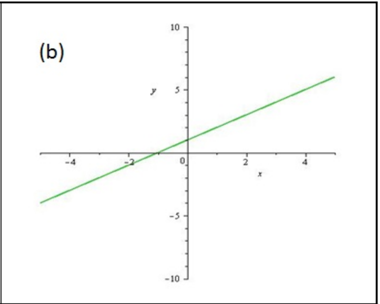
$$\{(x_1, x_2) \,|\, x_2 = 2x_1 + 1\}.$$

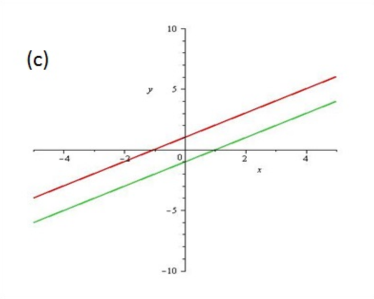
Notice that setting $x_2 = 2x_1 + 1$ in each equation we get the pair of true statements

$$2x_1 - (2x_1 + 1) = -1 \quad \text{and} \quad -4x_1 + 2(2x_1 + 1) = 2.$$

The Geometry of 2 Equations with 2 Variables







Graphical Illustration of Solution Cases

(a) $\begin{aligned} x - y &= -1 \\ 2x + y &= 3 \end{aligned}$ One solution case. Intersecting lines

(b) $\begin{aligned} x - y &= -1 \\ 2x - 2y &= -2 \end{aligned}$ Infinitely many solutions. One line

(c) $\begin{aligned} x - y &= -1 \\ 2x - 2y &= 2 \end{aligned}$ No solutions case. Parallel lines

3 Equations in 3 Variables

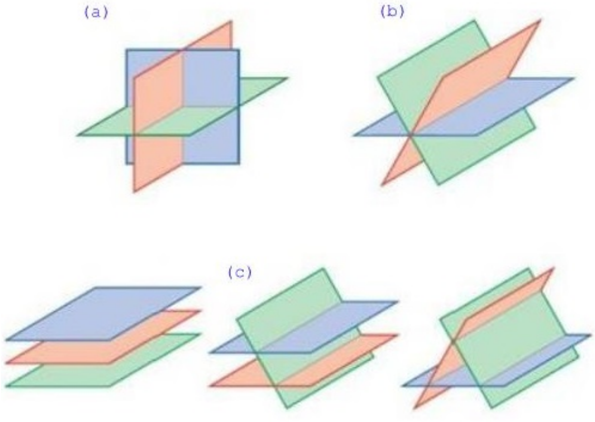


Figure: The graph of $ax + by + cz = d$ is a plane. Three may (a) intersect in a single point, (b) intersect in infinitely many points, or (c) not intersect in various ways.

Theorem

A linear system of equations has exactly one of the following:

i No solution, or

ii Exactly one solution, or

iii Infinitely many solutions.

Terms: A system is
consistent if it has at least one solution (cases ii and iii), and
inconsistent if is has no solutions (case i).

Note: This theorem speaks to those two big questions:

- Existence: Is there a solution/does a solution exist?
- Uniqueness: Is there a unique solution or multiple solutions?

Matrices

Definition: A matrix is a rectangular array of numbers. It's **size** (a.k.a. dimension/order) is $m \times n$ (read "m by n") where m is the number of rows and n is the number of columns the matrix has.

Examples:

$$\begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & 1 & 13 & -4 \\ 12 & -3 & 2 & -2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 0 \\ 4 & 4 \\ 3 & -5 \end{bmatrix}$$

3×4 $\qquad$ 3×2

Linear System & Matrices

Given any linear system of equations, we can associate two matrices with the system. These are the **coefficient** matrix and the **augmented** matrix.

Example:
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

Before we start to set up these matrices, we write our system in the form shown above. Note that all variables are on the left side, and like variables have the same order in each equation (they are aligned vertically).

Linear System: Coefficient Matrix

The **coefficient** matrix has one row for each equation and one column for each variable. The entries are the coefficients of the variables in our system.

Example:
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

$m = \#$ equations

$n = \#$ variable

$m = 3, \quad n = 3$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Linear System: Augmented Matrix

The **augmented** matrix has one row for each equation, one column for each variable, and one extra, right most column. The entries in the first columns match the coefficient matrix, and the right most column has the numbers from the right hand side of each equation.

Example:
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4\\ 2x_1 & & & + & x_3 & = & 7\\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

$m = \#$ of eqns

$n = \#$ variable $+ 1$

$m = 3, \; n = 4$

$$\begin{bmatrix} 1 & 2 & -1 & -4\\ 2 & 0 & 1 & 7\\ 1 & 1 & 1 & 6 \end{bmatrix}$$

Legitimate Operations for Solving a System

We can perform three basic operation **without** changing the solution set of a system. These are

- swap the order of any two equations **(swap)**,
- multiply an equation by any nonzero constant **(scale)**, and
- replace an equation with the sum[1] of itself and a nonzero multiple of any other equation **(replace)**.

We'll try to solve a system by using these operations to *eliminate* variables from equations.

[1] Adding equations means adding like variables.

Some Operation Notation

Notation

- Swap equations i and j:

$$E_i \leftrightarrow E_j$$

$i, j = 1, \dots, m$

- Scale equation i by k:

$$kE_i \rightarrow E_i$$

- Replace equation j with the sum of itself and k times equation i:

$$kE_i + E_j \rightarrow E_j$$

Example: Using operations to solve by elimination

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

We'll use the three operations (swap, scale, replace) to try to solve this system. Let's look at the augmented matrix at each step.

Punch line: This system can be made to look like

$$\begin{array}{rcl} x_1 & = & 1 \\ x_2 & = & 0 \\ x_3 & = & 5 \end{array}$$

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

We'll use x_1 in E_1 to eliminate x_1 from E_2 + E_3

Augmented matrix is
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & 6 \end{array}\right]$$

Perform

$-2E_1 + E_2 \to E_2$ and $-E_1 + E_3 \to E_3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -4x_2 + 3x_3 &= 15 \\ -x_2 + 2x_3 &= 10 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & -4 & 3 & 15 \\ 0 & -1 & 2 & 10 \end{array}\right]$$

Let's do $E_2 \leftrightarrow E_3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -x_2 + 2x_3 &= 10 \\ -4x_2 + 3x_3 &= 15 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -1 & 2 & 10 \\ 0 & -4 & 3 & 15 \end{bmatrix}$$

Use E_2 to eliminate x_2 from E_3

$$-4E_2 + E_3 \to E_3$$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -x_2 + 2x_3 &= 10 \\ -5x_3 &= -25 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -1 & 2 & 10 \\ 0 & 0 & -5 & -25 \end{bmatrix}$$

Let's scale $-\frac{1}{5}E_3 \to E_3$ and $-E_2 \to E_2$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ x_2 - 2x_3 &= -10 \\ x_3 &= 5 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & 1 & -2 & -10 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Use E_3 to eliminate x_3 from E_1 and E_2

$2E_3 + E_2 \to E_2 \qquad E_3 + E_1 \to E_1$

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ x_2 &= 0 \\ x_3 &= 5 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Eliminate x_2 from E_1 using

$-2E_2 + E_1 \to E_1$

$$\begin{aligned} x_1 \qquad\qquad &= 1 \\ x_2 \qquad &= 0 \\ x_3 &= 5 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

The system has one solution $(1, 0, 5)$.