January 12 Math 3260 sec. 52 Spring 2024

Last time, we saw that some **equivalent systems** had augmented matrices

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -1 & 2 & 10 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

These are all **row equivalent** matrices. Regarding the underlying system, from

- the first, it's not obvious is the system is consistent,
- the second, it's clearly consistent and it would be easy to find the solution with a little back substitution, and
- the third, it's obviously consistent and the solution is obvious.

The latter two have a nice structure.

Section 1.2: Row Reduction and Echelon Forms

Definition

A matrix is in **echelon form**, also called *row echelon form (ref)*, if it has the following properties:

i Any row of all zeros are at the bottom.

ii The first nonzero number (called the *leading entry*) in a row is to the right of the first nonzero number in all rows above it.

iii All entries below a leading entry are zeros.[a]

[a] This condition is superfluous but is included for clarity.

an ref

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$$

not an ref

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Reduced Echelon Form

Definition

A matrix is in **reduced echelon form**, also called *reduced row echelon form (rref)* if it is in echelon form and has the additional properties

iv The leading entry of each row is 1 (called a *leading* 1), and

v each leading 1 is the only nonzero entry in its column.

an rref

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

not an rref

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Refs and Rrefs

Identify each matrix as being an echelon form (ref), reduced echelon form (rref) or not an echelon form.

(a) $\begin{bmatrix} -1 & 2 & 3 & 0 & 1 \\ 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ref but not rref

not an echelon form

(c) $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

rref

ref not an rref

Example (finding ref's and rref's)

Find an echelon form for the following matrix using elementary row operations.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 0 & 3 & 2 \end{bmatrix}$$

$-2R_1 + R_2 \to R_2$

$$\begin{matrix} -4 & -2 & -6 \\ 4 & 3 & 6 \end{matrix}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix}$$

$-3R_2 + R_3 \to R_3$

$$\begin{matrix} 0 & -3 & 0 \\ 0 & 3 & 2 \end{matrix}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

this is an ref