January 13 Math 2306 sec. 52 Spring 2023

Section 1: Concepts and Terminology

We

- defined *differential equation* along with classifying variables as dependent or independent,
- classified diff. eqs. as *ordinary* (ODE) or *partial* (PDE),
- defined the *order* of a diff. eq.
- defined *linear* differential equations (as compared to nonlinear ones),
- and defined a *solution* (or explicit solution)

Solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ (*)

Solution or Explicit Solution

Definition: A function ϕ defined on an interval[1] I and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Example: We verified that $\phi(x) = c_1 x + \dfrac{c_2}{x}$ is a solution to the ODE $x^2 y'' + xy' - y = 0$ for any choice of the constants c_1 and c_2 on the interval $I = (0, \infty)$ because

- it is twice differentiable, and
- when we substituted $y = c_1 x + \dfrac{c_2}{x}$ into the equation, it resulted in a true statement.

[1] The interval is called the *domain of the solution* or the *interval of definition*.

Example:

Verify that $\phi(t) = \dfrac{1}{t}$ is a solution to the differential equation

$$\frac{dy}{dt} = -y^2,$$

and explain why the interval of definition for this solution can be $(0, 1)$, but it can't be $(-1, 1)$.

$\phi(t) = \frac{1}{t}$ is differentiable on any interval not containing zero.

Set $y = \phi$, $y = \frac{1}{t}$.

Evaluate the left hand side:

$$\frac{dy}{dt} = \frac{d}{dt} t^{-1} = -t^{-2} = \frac{-1}{t^2}$$

Evaluate the right hand side:

$$-y^2 = -\left(\frac{1}{t}\right)^2 = \frac{-1}{t^2}$$

so for $y = \frac{1}{t}$, $\frac{dy}{dt} = \frac{-1}{t^2} = -y^2$.

$(-1,1)$ can't be the domain because $y = \frac{1}{t}$ isn't defined or differentiable at $t=0$.

$\phi(t) = \frac{1}{t}$ is differentiable on $(0,1)$.

Solution of $F(x, y, y', \dots, y^{(n)}) = 0$ (*)

Implicit Solution

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

Recall that a **relation** is an equation in the two variables x and y. Something like

$$x^2 + y^2 = 4, \quad \text{or} \quad xy = e^y$$

would be examples of relations.

Example: Implicitly Defined Solution(s)

Verify that the relation(left) defines and implicit solution of the differential equation (right).

$$y^2 - 2x^2y = 1, \qquad \frac{dy}{dx} = \frac{2xy}{y - x^2}$$

We'll show that when the relation is true, the ODE is also true. We'll use implicit differentiation.

$$\frac{d}{dx}\left(y^2 - 2x^2y\right) = \frac{d}{dx}(1)$$

$$2y\frac{dy}{dx} - 2\left(2xy + x^2\frac{dy}{dx}\right) = 0$$

It may not be possible to clearly identify the domain of definition of an implicit solution.

Isolate $\frac{dy}{dx}$.

$$y\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} - x^2\frac{dy}{dx} = 2xy$$

$$(y - x^2)\frac{dy}{dx} = 2xy$$

$$\frac{dy}{dx} = \frac{2xy}{y - x^2}, \quad \text{for } y - x^2 \neq 0$$

This shows that $y^2 - 2x^2y = 1$ defines a solution to $\frac{dy}{dx} = \frac{2xy}{y - x^2}$.

Function vs Solution

The interval of defintion has to be an interval.

Consider the ODE

$$\frac{dy}{dx} = -y^2.$$

The function $y = \frac{1}{x}$ is a solution. The domain of $f(x) = \frac{1}{x}$

- as a **function** could be stated as $(-\infty, 0) \cup (0, \infty)$.
- as a **solution** to an ODE could be stated as $(0, \infty)$, or as $(-\infty, 0)$.

In the absence of additional information, we'll usually take the interval of definition to be the largest possible one (or one of the largest possible ones).

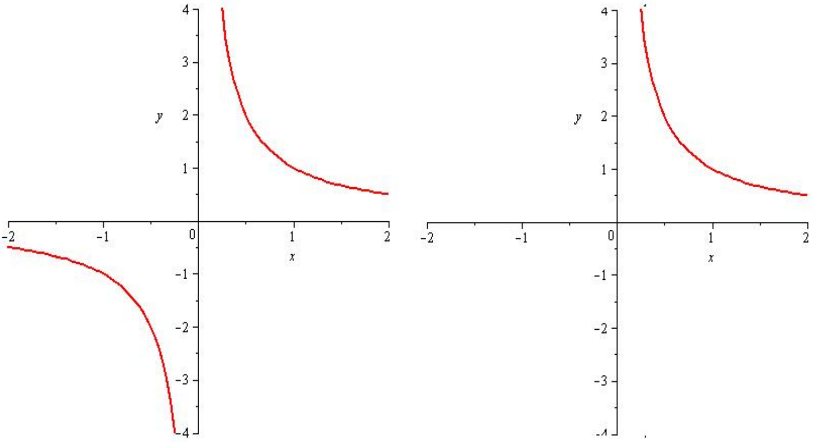

Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function**. Right: Plot of $f(x) = \frac{1}{x}$ as a possible **solution** of an ODE.

Systems of ODEs

Sometimes we want to consider two or more dependent variables that are functions of the same independent variable. The ODEs for the dependent variables can depend on one another. Some examples of relevant situations are

- predator and prey
- competing species
- two or more masses attached to a system of springs
- two or more composite fluids in attached tank systems

Such systems can be **linear** or **nonlinear**.

Example of Nonlinear System

$$\begin{aligned}\frac{dx}{dt} &= -\alpha x + \beta xy\\ \frac{dy}{dt} &= \gamma y - \delta xy\end{aligned}$$

This is known as the **Lotka-Volterra** predator-prey model. $x(t)$ is the population (density) of predators, and $y(t)$ is the population of prey. The numbers α, β, γ and δ are nonnegative constants.
This model is built on the assumptions that

- in the absence of predation, prey increase exponentially
- in the absence of predation, predators decrease exponentially,
- predator-prey interactions increase the predator population and decrease the prey population.

Example of a Linear System

$$\begin{aligned}\frac{di_2}{dt} &= -2i_2 - 2i_3 + 60 \\ \frac{di_3}{dt} &= -2i_2 - 5i_3 + 60\end{aligned}$$

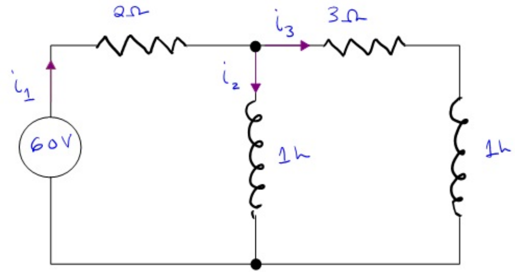

Figure: Electrical Network of resistors and inductors showing currents i_2 and i_3 modeled by this system of equations.

Solution of a System

When we talk about a **solution** to a system of ODEs, we mean a set of functions, one for each dependent variable. For example, a solution to

$$\begin{aligned}\frac{di_2}{dt} &= -2i_2 - 2i_3 + 60\\ \frac{di_3}{dt} &= -2i_2 - 5i_3 + 60\end{aligned}$$

would have to include functions for both of i_2 and i_3.

A fun exercise is to show that

$$\begin{aligned}i_2(t) &= 30 - 24e^{-t} - 6e^{-t}\\ i_3(t) &= 12e^{-t} - 12e^{-6t}\end{aligned}$$

gives a solution. This is what you get if you assume that the initial currents are all zero.

Systems of ODEs

There are various approaches to solving a system of differential equations. These can include

- elimination (try to eliminate a dependent variable),
- matrix techniques,
- Laplace transforms[2]
- numerical approximation techniques

[2] We'll consider this later.

Some Terms

- A **parameter** is an unspecified constant (such as c_1 and c_2 in the last example with $\phi(x) = c_1 x + \frac{c_2}{x}$).
- A **family of solutions** is a collection of solution functions that only differ by a parameter.
- An ***n*-parameter family of solutions** is one containing n parameters (e.g. $\phi(x) = c_1 x + \frac{c_2}{x}$ is a 2 parameter family).
- A **particular solution** is one with no arbitrary constants in it.
- The **trivial solution** is the simple constant function $y = 0$.
- An **integral curve** is the graph of one solution (perhaps from a family).