January 14 Math 3260 sec. 52 Spring 2022

Section 1.1: Systems of Linear Equations

Recall that a **linear system** of (algebraic) equations in n variables $x_1, \dots, x_n$ is one of the form

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1\\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2\\
 & \vdots & & & \vdots & & \vdots & & \vdots\\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m.
\end{array}$$

Theorem on Solutions

A linear system of equations has exactly one of the following:

i No solution, or

ii Exactly one solution, or

iii Infinitely many solutions.

We said that a system is

- **inconsistent** if it has no solutions and
- **consistent** if it has at least one solution.

Linear Systems & Matrices

Given a linear system, we can identify two matrices corresponding to that system.

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array}$$

The **coefficient matrix** and the **augmented matrix**.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Elementary Row Operations: On a Matrix

Definition

We have three operations we can perform on a matrix that are called **Elementary Row Operations**.

i Interchange any two rows (**row swap**).

ii Multiply a row by any nonzero constant (**scaling**).

iii Replace a row with the sum of itself and a multiple of another row (**replacement**).

Definition: If any sequence of elementary row operations are performed on a matrix, the resulting matrix is called **row equivalent**.

to the original matrix

Theorem on Row Equivalent Matrices

Theorem: If the augmented matrices of two linear systems are row equivalent, then the systems have the same solution set. That is, the linear systems are equivalent.

Augmented Matrix

We saw the example

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \qquad \begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

Through a sequence of operation (swapping equations, scaling equations, and replacing equations), we transformed this system into the **equivalent** system

$$\begin{array}{ccccl} x_1 & & & = & 1 \\ & x_2 & & = & 0 \\ & & x_3 & = & 5 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

A key here is *structure*!

Consider the following augmented matrix. Determine if the associated system is consistent or inconsistent. If it is consistent, determine the solution set.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$

the system is

$$1x_1 + 0x_2 + 0x_3 = 3$$
$$0x_1 + 1x_2 + 0x_3 = 1$$
$$0x_1 + 0x_2 + 1x_3 = -2$$

This says $x_1 = 3$, $x_2 = 1$, $x_3 = -2$.

The system is consistent and has one solution $(3, 1, -2)$. We can also state this as

$$x_1 = 3$$
$$x_2 = 1$$
$$x_3 = -2$$

(b) $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

the system is

$$1x_1 + 2x_2 + 0x_3 = 3$$
$$0x_1 + 1x_2 - 1x_3 = 4$$
$$0x_1 + 0x_2 + 0x_3 = 3$$

this last equation says

$$0 = 3$$

which is always false.

The system is inconsistent.

(c) $\begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

the system is

$$1x_1 + 0x_2 - 2x_3 = -3$$
$$0x_1 + 1x_2 + 1x_3 = 4$$
$$0x_1 + 0x_2 + 0x_3 = 0$$

The last equation is $0=0$ which is always true.

The two top equations are
$$x_1 - 2x_3 = -3$$
$$x_2 + x_3 = 4$$

which we can write as

$x_1 = -3 + 2x_3$, x_3 can be any real number.
$x_2 = 4 - x_3$

This system is consistent and has infinitely many solutions where $x_1 = -3 + 2x_3$, $x_2 = 4 - x_3$ and x_3 is any real number.