January 17 Math 3260 sec. 52 Spring 2024

Section 1.2: Row Reduction and Echelon Forms

Echelon (ref) and Reduced Echelon (rref) Forms

We said that a matrix is an **echelon form (ref)** if

i any rows consisting of all zeros are at the bottom of the matrix, and

ii the leading entry[a] in any row is to the right of all leading entries in the rows above it—i.e., all the terms below a leading entry are zero.

Moreover, an echelon form is called a **reduced echelon form (rref)** if, in addition

i every leading entry is a 1, and

ii each leading[b] 1 is the only nonzero entry in its column.

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[a] A leading entry is the leftmost nonzero entry in a row.

[b] A leading entry that is a 1 is called a leading one.

Example (finding ref's and rref's)

Find an echelon form for the following matrix using elementary row operations.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 0 & 3 & 2 \end{bmatrix}$$

We were working on this problem. We found an ref in two steps.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 0 & 3 & 2 \end{bmatrix} \quad -2R_1 + R_2 \to R_2 \quad \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix} \quad -3R_2 + R_3 \to R_3 \quad \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Example (finding ref's and rref's)

After this second step, we have an ref: $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Remark: We could make other choices and obtain a different **row equivalent** matrix that is also an ref. For example, we could scale the first and third rows by one half to get

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \begin{array}{c} \frac{1}{2}R_1 \to R_1 \\ \frac{1}{2}R_3 \to R_3 \end{array} \qquad \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Both of these matrices are echelon forms, and both are row equivalent to the original matrix. This tells us that

$$\boxed{\text{refs are NOT unique.}}$$

Example (rref)

Find the reduced echelon form for the following matrix.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 0 & 3 & 2 \end{bmatrix}$$

We'll start with the ref we already found.

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \frac{1}{2}R_3 \to R_3 \quad \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-3R_3 + R_1 \to R_1 \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-R_2 + R_1 \to R_1$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \frac{1}{2}R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is the rref

Pivot Positions & Pivot Columns

Theorem

The reduced row echelon form of a matrix is unique.

That is, a given matrix is row equivalent to many different refs but to only ONE rref! This allows for the following unambiguous definitions.

Pivot Position & Pivot Column

Definition: A **pivot position** in a matrix A is a location that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

Identifying Pivot Positions and Columns

The following matrices are **row equivalent**. Identify the pivot positions and pivot columns of the matrix A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

B is an rref.

From the rref, the pivot positions in A (and B) are row 1 column 1, row 2 column 2, and row 3 column 4.

The pivot columns are 1, 2 and 4.

Complete Row Reduction isn't needed to find Pivots

The following three matrices are row equivalent. (Note, B is an ref but not an rref, and C is an rref.)

$$A = \begin{bmatrix} 1 & 1 & 4 \\ -2 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Identify the pivot positions and pivot columns of the matrix A.

From B, we see that pivot positions are row 1 column 1, and row 2 column 2.

Pivot columns are 1 and 2.

Row Reduction Algorithm

To obtain an echelon form, we work from left to right beginning with the top row working downward.

$$\begin{bmatrix} 0 & 3 & -6 & 4 & 6 \\ 3 & -7 & 8 & 8 & -5 \\ 3 & -9 & 12 & 6 & -9 \end{bmatrix} \qquad (R_1 \leftrightarrow R_3)$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Step 1: The left most column is a pivot column. The top position is a pivot position.

Step 2: Get a nonzero entry in the top left position by row swapping if needed.

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

$-R_1 + R_2 \to R_2$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Step 3: Use row operations to get zeros in all entries below the pivot.

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Choices:

$-\frac{3}{2}R_2 + R_3 \to R_3$

or

$\frac{1}{2}R_2 \to R_2$ and

$-3R_2 + R_3 \to R_3$

$\frac{1}{2}R_2 \to R_2$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 3 & -6 & 4 & \end{bmatrix}$$

$-3R_2 + R_3 \to R_3$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Step 4: Ignore the row with a pivot, all rows above it, the pivot column, and all columns to its left, and repeat steps 1-3.

Row Reduction Algorithm

we have an ref

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Row Reduction Algorithm

To obtain a reduced row echelon form:

Step 5: Starting with the right most pivot and working up and to the left, use row operations to get a zero in each position above a pivot. Scale to make each pivot a 1.

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$-R_3 + R_2 \to R_2$

$-6R_3 + R_1 \to R_1$

$$\begin{bmatrix} 3 & -9 & 12 & 0 & -9 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$9R_2 + R_1 \to R_1$

$0 \quad 9 \quad -18 \quad 0 \quad 18$

$3 \quad -9 \quad 12 \quad 0 \quad -9$

Row Reduction Algorithm

$$\begin{bmatrix} 3 & 0 & -6 & 0 & 9 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \frac{1}{3}R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

the rref.

Echelon Form & Solving a System

Recall: Row equivalent matrices correspond to equivalent systems.

Suppose the matrix on the left is the augmented matrix for a linear system of equations in the variables x_1, x_2, x_3 and x_4. Use the rref to characterize the solution set to the linear system.

$$\begin{bmatrix} 0 & 3 & -6 & 4 & 6 \\ 3 & -7 & 8 & 8 & -5 \\ 3 & -9 & 12 & 6 & -9 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad - 2x_3 \quad &= 3 \\ x_2 - 2x_3 \quad &= 2 \\ x_4 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= 3 + 2x_3 \\ x_2 &= 2 + 2x_3 \\ x_4 &= 0 \end{aligned}$$

x_3 is a variable

Basic & Free Variables

Definition

Suppose a system has m equations and n variables, $x_1, x_2, \ldots, x_n$. The first n columns of the augmented matrix correspond to the n variables. For each i such that $1 \leq i \leq n$:

- If the i^{th} column is a pivot column, then x_i is called a **basic variable**.
- If the i^{th} column is NOT a pivot column, then x_i is called a **free variable**.

Basic & Free Variables

Consider the system of equations along with its augmented matrix.

$$\begin{array}{rcrcrcrcr} & & 3x_2 & - & 6x_3 & + & 4x_4 & = & 6 \\ 3x_1 & - & 7x_2 & + & 8x_3 & + & 8x_4 & = & -5 \\ 3x_1 & - & 9x_2 & + & 12x_3 & + & 6x_4 & = & -9 \end{array} \qquad \begin{bmatrix} 0 & 3 & -6 & 4 & 6 \\ 3 & -7 & 8 & 8 & -5 \\ 3 & -9 & 12 & 6 & -9 \end{bmatrix}$$

We determined that the matrix was row equivalent to the rref

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Identify the free and basic variables.

Basic Variable(s): x_1, x_2, x_4

Free Variable(s): x_3

Expressing Solutions

To avoid confusion, i.e., in the interest of clarity, we will **always** write solution sets by expressing basic variables in terms of free variables. We will not write free variables in terms of basic. That is, the solution set to the system whose augmented matrix is row equivalent to

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

will be written

$$\begin{aligned} x_1 &= 3 + 2x_3 \\ x_2 &= 2 + 2x_3 \\ x_3 & \quad \text{is free} \\ x_4 &= 0 \end{aligned}$$

This is called a *parametric* form or description of the solution set.

Proper Solution Set Expressions

We will never express free variables in terms of basic variables. All three of the following result from the same augmented matrix:

$$\begin{array}{lcl} x_1 & = & 3+2x_3 \\ x_2 & = & 2+2x_3 \\ x_3 & & \text{is free} \\ x_4 & = & 0 \end{array} \qquad \begin{array}{lcl} x_3 & = & 3/2+1/2x_1 \\ x_2 & = & 2+2x_3 \\ x_3 & & \text{is free} \\ x_4 & = & 0 \end{array} \qquad \begin{array}{lcl} x_1 & = & 3+2x_3 \\ x_2 & = & 1+x_1 \\ x_3 & & \text{is free} \\ x_4 & = & 0 \end{array}$$

The left most parametric description is correct. The two expressions in red are **not correct** descriptions—even though they all follow from the same matrix!

Consistent versus Inconsistent Systems

Consider each ~~r~~ref. Determine if the underlying system (the one with this as its augmented matrix) is consistent or inconsistent.

(a) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Consistent, ∞-many solutions

(b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ Consisten one solution

(c) $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ inconsistent

An Existence and Uniqueness Theorem

Theorem

A linear system is consistent if and only if the right most column of the augmented matrix is **NOT** a pivot column. That is, if and only if each echelon form **DOES NOT** have a row of the form

$$[0\ 0\ \cdots\ 0\ b], \quad \text{for some nonzero } b.$$

Moreover, if a linear system is consistent, then it has

(i) exactly one solution if there are no free variables, and

(ii) infinitely many solutions if there is at least one free variable.