January 18 Math 2306 sec. 51 Spring 2023

Section 2: Initial Value Problems

An **initial value problem (IVP)** consists of an ODE

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$$

coupled with a set if **initial conditions (IC)**

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, y^{(n-1)}(x_0) = y_{n-1}.$$

Note two key features of the problem

- The number of ICs matches the order of the ODE, and
- all IC are given at the same specified input value x_0.

IVPs

First order case:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

- The ODE gives info on the shape of a solution.

- The IC indicates that the solution must pass through the point (x_0, y_0)

IVPs

Second order case:

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

If y is the position of a particle moving along a line

y'' gives the acceleration @ time x.

y_0 is the initial position

and y_1 is the initial velocity.

System of IVPs

We can also consider a system of IVP for example

$$\begin{aligned}\frac{di_2}{dt} &= -2i_2 - 2i_3 + 60, \quad i_2(t_0) = i_{2_0}\\ \frac{di_3}{dt} &= -2i_2 - 5i_3 + 60, \quad i_3(t_0) = i_{3_0}\end{aligned}$$

The number of initial conditions for each dependent variable will match the highest order derivative for *that* dependent variable. All initial conditions for all dependent variables are given at the same input value (t_0).

Example

Given that $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family of solutions of $x^2 y'' + xy' - y = 0$, solve the IVP

$$x^2 y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

The solutions to the ODE are in the family $y = c_1 x + \frac{c_2}{x}$. We have to find the values of c_1 and c_2 so that the I C are satisfied.

$y = c_1 x + \frac{c_2}{x}$, $\quad y' = c_1 - \frac{c_2}{x^2}$

Apply $y(1) = 1$ $\quad y(1) = c_1(1) + \frac{c_2}{1} = 1$

Apply $y'(1) = 3$ $\quad y'(1) = c_1 - \frac{c_2}{1^2} = 3$

We need to solve

$$c_1 + c_2 = 1$$
$$c_1 - c_2 = 3$$

add $\quad 2c_1 = 4 \implies c_1 = 2$

From the 1^{st} eqn. $\quad c_2 = 1 - c_1 = 1 - 2 = -1$

The solution to the IVP is

$$\boxed{y = 2x - \frac{1}{x}}$$

$y = c_1 x + \frac{c_2}{x}$ is a 2 parameter family of solutions to

$$x^2 y'' + x y' - y = 0$$

$y = 2x - \frac{1}{x}$ is a particular solution to the ODE and it is the solution to the IVP.