January 19 Math 3260 sec. 52 Spring 2022

Section 1.2: Row Reduction and Echelon Forms

Recall the examples of augmented matrices for which is was easy to determine whether the system was consistent or inconsistent and what solutions it might have.

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Some key features we can look for are

- Columns having only one nonzero entry that is a 1, and
- A row with all zeroes or all zeros except for the last entry which in nonzero.

Row Reduction and Echelon Forms

Definition:

A matrix is in **echelon form** (a.k.a. **row echelon form**) if the following properties hold

ref

i Any row of all zeros are at the bottom.

ii The first nonzero number (called the *leading entry*) in a row is to the right of the first nonzero number in all rows above it.

iii All entries below a leading entry are zeros.

Is

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$$

Is Not

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Reduced Echelon Form

Definition: A matrix is in **reduced echelon form** (a.k.a. **reduced row echelon form**) if it is in echelon form and the following additional properties hold

rref

iv The leading entry of each row is 1 (called a *leading* 1), and

v each leading 1 is the only nonzero entry in its column.

Is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Is Not

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Notation for Elementary Row Operations

The **Elementary Row Operations** are

i Interchange any two rows (**row swap**).

Swap rows i and j: $\quad R_i \leftrightarrow R_j$

ii Multiply a row by any nonzero constant (**scaling**).

Scale row i by k: $\quad kR_i \rightarrow R_i$

iii Replace a row with the sum of itself and a multiple of another row (**replacement**).

Replace row j with the sum of itself and k times row i:

$$kR_i + R_j \rightarrow R_j$$

Example (finding ref's and rref's)

Find an echelon form for the following matrix using elementary row operations.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 0 & 3 & 2 \end{bmatrix}$$

- want nonzero entry in row 1 column 1.
- want all zero below that leading entry.

Do $-2R_1 + R_2 \to R_2$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix}$$

scratch

$$\begin{matrix} -4 & -2 & -6 \\ 4 & 3 & 6 \end{matrix}$$

- Now we want a nonzero entry in row 2 Column 2.
- We want zeros below it.

Do $-3R_2 + R_3 \to R_3$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

This is an ref.

Example (rref)

Find the reduced echelon form for the following matrix.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 0 & 3 & 2 \end{bmatrix}$$

We can start with the ref we just found.

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- Need each leading entry to be 1
- Need each leading 1 to be the only nonzero entry in its column.

Scale R_3, $\frac{1}{2}R_3 \to R_3$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To clear out the third column do $-3R_3 + R_1 \to R_1$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Clear out column 2

$-R_2 + R_1 \to R_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally, $\frac{1}{2} R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is an rref.

Theorem

Theorem: Each matrix is row equivalent to exactly one reduced row echelon matrix.

This says that the rref of a matrix is unique, and it allows the following unambiguous definition:

Definition: A **pivot position** in a matrix A is a location that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** of a matrix A is a column of A that contains a pivot position.