January 20 Math 2306 sec. 51 Spring 2023

Section 2: Initial Value Problems

Recall: an **initial value problem (IVP)** consists of an ODE

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

coupled with a set if **initial conditions (IC)**

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, y^{(n-1)}(x_0) = y_{n-1}.$$

A first order IVP looks like

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

A Numerical Solution

Consider a first order initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Euler's Method is a scheme for finding an approximate solution to this IVP. The basic idea is that we

- ▶ Start with the known point (x_0, y_0) on the solution curve,
- ▶ use the slope (given by $\frac{dy}{dx}$) to get a tangent line there, and
- ▶ approximate a nearby point on the curve by the tangent line.
- ▶ march forward a littel bit, and repeat.

Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$

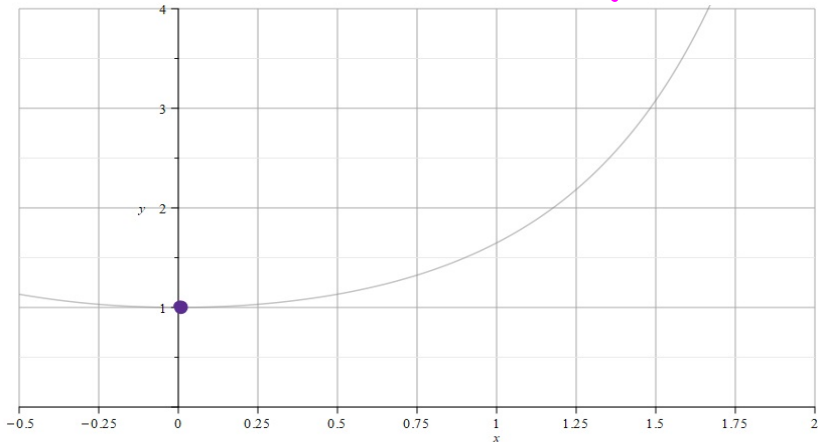*[handwritten annotations: $f(x,y)$ pointing to xy; $x_0 = 0$, $y_0 = 1$]*

Figure: We know that the point $(x_0, y_0) = (0, 1)$ is on the curve. And the slope of the curve at $(0, 1)$ is $m_0 = f(0, 1) = 0 \cdot 1 = 0$.

Note: The gray curve is the true solution to this IVP. It's shown for reference.
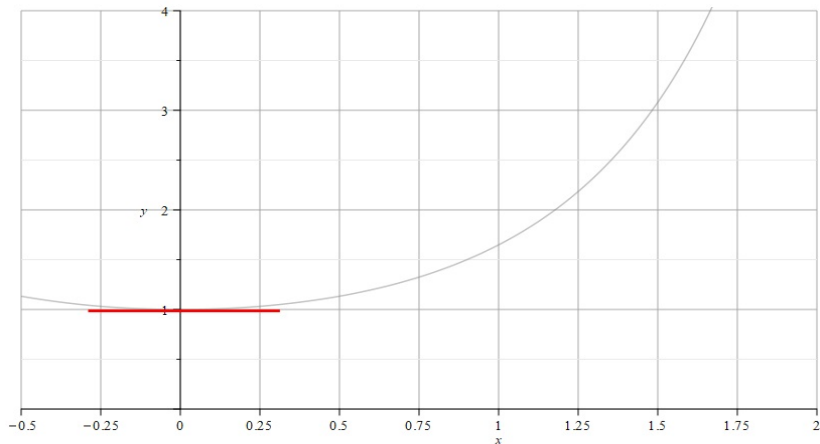
Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

Figure: So we draw a little tangent line (we know the point and slope). Then we increase x, say $x_1 = x_0 + h$, and approximate the solution value $y(x_1)$ with the value on the tangent line y_1. So $y_1 \approx y(x_1)$.

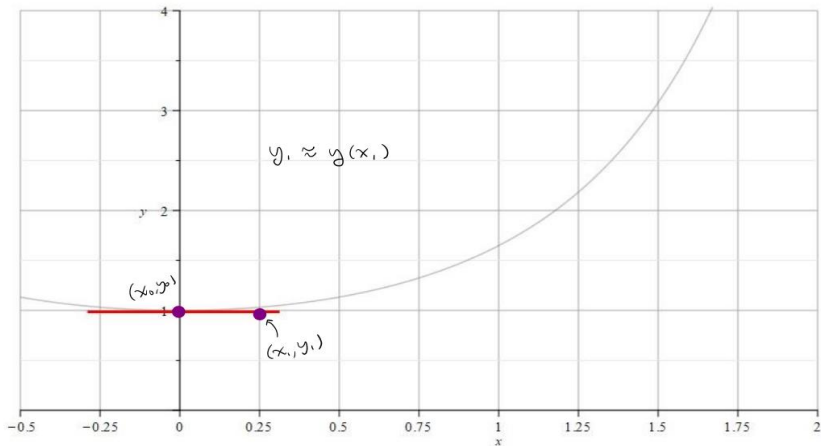Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$

Figure: We take the approximation to the true function y at the point $x_1 = x_0 + h$ to be the point on the tangent line.

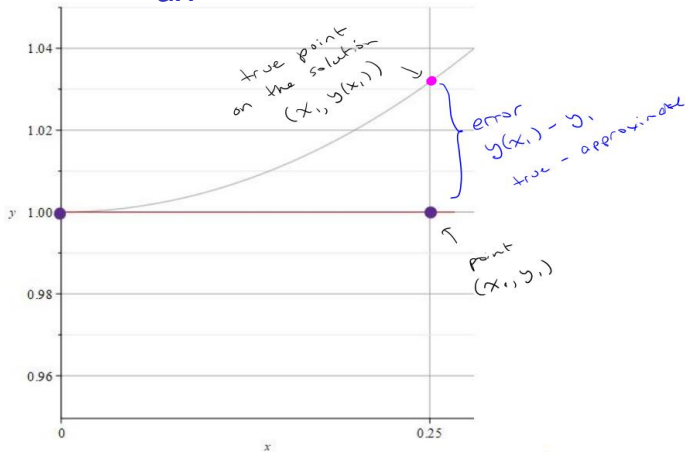Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$

Figure: When *h* is very small, the true solution and the tangent line point will be close. Here, we've zoomed in to see that there is some error between the exact *y* value and the approximation from the tangent line.

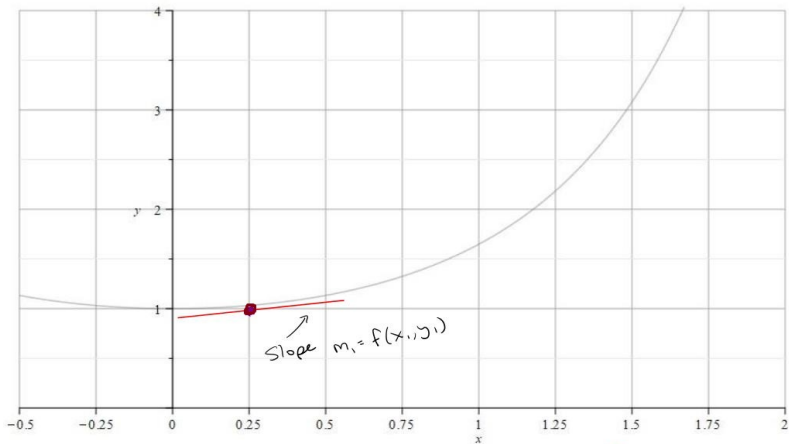Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$

Figure: Now we start with the point (x_1, y_1) and repeat the process. We get the slope $m_1 = f(x_1, y_1)$ and draw a tangent line through (x_1, y_1) with slope m_1.

Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$
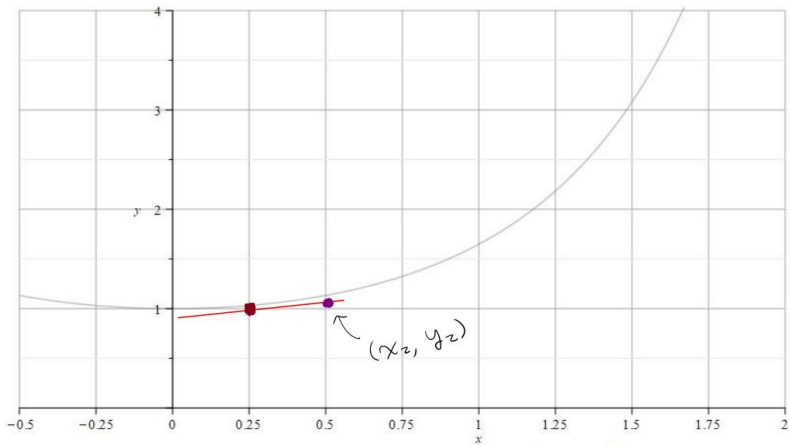

Figure: We go out h more units to $x_2 = x_1 + h$. Pick the point on the tangent line (x_2, y_2), and use this to approximate $y(x_2)$. So $y_2 \approx y(x_2)$

Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$
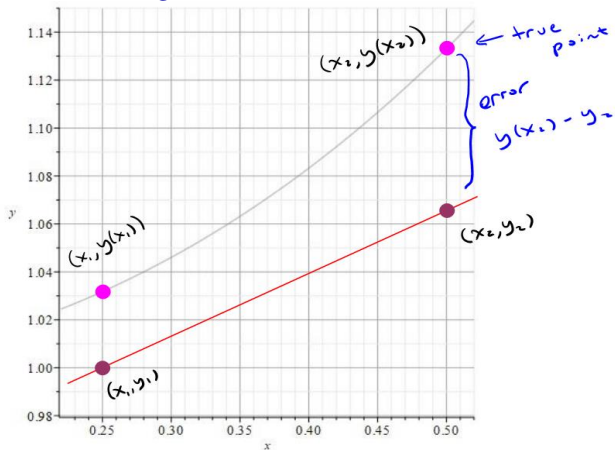
Figure: If we zoom in, we can see that there is some error. But as long as *h* is small, the point on the tangent line approximates the point on the actual solution curve.

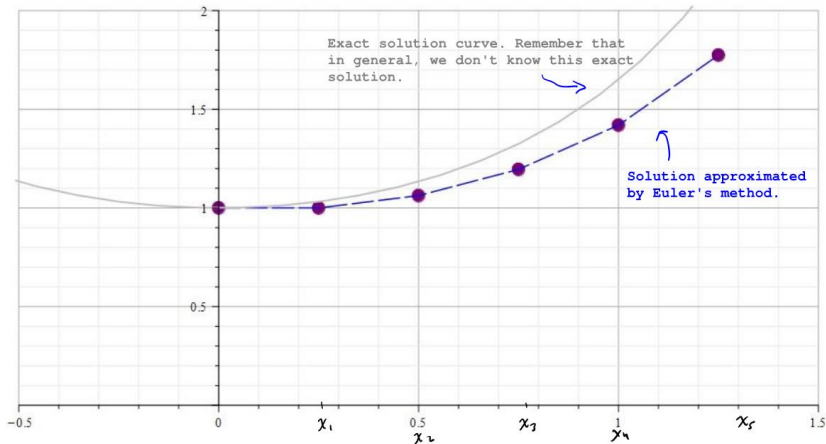Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$

Figure: We can repeat this process at the new point to obtain the next point. We build an approximate solution by advancing the independent variable and connect the points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$.

Euler's Method: An Algorithm & Error

We start with the IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

We build a sequence of points that approximates the true solution y

$$(x_0, y_0), \ (x_1, y_1), \ (x_2, y_2), \ \ldots, (x_N, y_N).$$

We'll take the x values to be equally spaced with a common difference of h. That is

$$
\begin{aligned}
x_1 &= x_0 + h \\
x_2 &= x_1 + h = x_0 + 2h \\
x_3 &= x_2 + h = x_0 + 3h \\
&\vdots \\
x_n &= x_0 + nh
\end{aligned}
$$

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Notation:

- y_n will denote our approximation, and

- $y(x_n)$ will denote the exact solution (that we don't know)

To build a formula for the approximation y_1, let's approximate the derivative at (x_0, y_0).

$$f(x_0, y_0) = \left. \frac{dy}{dx} \right|_{(x_0, y_0)} \approx \frac{y_1 - y_0}{x_1 - x_0}$$

(Notice that's the standard formula for slope.)

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Let's get a formula for y_1.

$$\frac{y_1 - y_0}{x_1 - x_0} = f(x_0, y_0)$$

recall $\quad x_1 - x_0 = h$

$$\frac{y_1 - y_0}{h} = f(x_0, y_0) \Rightarrow y_1 - y_0 = h f(x_0, y_0)$$

Hence $\qquad y_1 = y_0 + h f(x_0, y_0)$

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

We can continue this process. So we use

$$\frac{y_2 - y_1}{h} = f(x_1, y_1) \quad \implies \quad y_2 = y_1 + hf(x_1, y_1)$$

and so forth. We have

Euler's Method Formula: The nth approximation y_n to the exact solution $y(x_n)$ is given by

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

with (x_0, y_0) given in the original IVP and h the choice of step size.

Euler's Method Example: $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

Take $h = 0.25$ to find an approximation to $y(1)$.

Here $f(x,y) = xy$, $x_0 = 0$, $y_0 = 1$

$x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$

We'll stop with $y_4 \approx y(1)$

$y_1 = y_0 + h f(x_0, y_0)$

$\qquad = 1 + 0.25 \, (0 \cdot 1) = 1$

$x_1 = 0.25, \quad y_1 = 1$

$$y_2 = y_1 + h \cdot f(x_1, y_1)$$
$$= 1 + 0.25 (0.25 \cdot 1) = 1.0625$$

$$x_2 = 0.5, \quad y_2 = 1.0625$$

$$y_3 = y_2 + h f(x_2, y_2)$$
$$= 1.0625 + 0.25 (0.5 \cdot 1.0625)$$
$$= 1.1953125$$

$$x_3 \quad .75, \quad y_3 = \quad 153125$$

$$y_4 = y_3 + h f(x_3, y_3)$$

$$= 1.1953125 + 0.25 \left(0.75 \cdot 1.1953125 \right)$$

$$= 1.419436$$

$$y_4 = 1.419436 \approx y(1)$$

Euler's Method Example: $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

Take $h = 0.25$ to find an approximation to $y(1)$.

We went through this process and found that $y_4 = 1.41943$ was our approximation to $y(1)$.

The true[1] $y(1) = \sqrt{e} = 1.64872$. This raises the question of how good our approximation can be expected to be.

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[1]The exact solution $y = e^{x^2/2}$.

Euler's Method: Error

First, let's define what we mean by the term *error*. There are a couple of types of error that we can talk about. These are[2]

$$\text{Absolute Error} = |\text{True Value} - \text{Approximate Value}|$$

and

$$\text{Relative Error} = \frac{\text{Absolute Error}}{|\text{True value}|}$$

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[2]Some authors will define absolute error without use of absolute value bars so that absolute error need not be nonnegative.

Euler's Method: Error

We can ask, how does the error depend on the step size?

$$\frac{dy}{dx} = xy, \quad y(0) = 1$$

I programed Euler's method into Matlab and used different *h* values to approximate $y(1)$, and recorded the results shown in the table.

h	$y(1) - y_n$	$\frac{y(1)-y_n}{y(1)}$
0.2	0.1895	0.1149
0.1	0.1016	0.0616
0.05	0.0528	0.0320
0.025	0.0269	0.0163
0.0125	0.0136	0.0082

Euler's Method: Error

We notice from this example that cutting the step size in half, seems to cut the error and relative error in half. This suggests the following:

> The absolute error in Euler's method is proportional to the step size.

There are two sources of error for Euler's method (not counting numerical errors due to machine rounding).

- ▶ The error in approximating the curve with a tangent line, and
- ▶ using the approximate value y_{n-1} to get the slope at the next step.

Euler's Method: Error

For numerical schemes of this sort, we often refer to the *order* of the scheme. If the error satisfies

$$\text{Absolute Error} = Ch^p$$

where C is some constant, then the order of the scheme is p.

Euler's method is an order 1 scheme.