January 21 Math 3260 sec. 51 Spring 2022

We defined the **row echelon form** and **reduced row echelon forms** of matrices.

Definition: A matrix is in **row echelon form** (ref) if the following properties hold

- i Any row of all zeros are at the bottom.
- ii The first nonzero number (called the *leading entry*) in a row is to the right of the first nonzero number in all rows above it.
- iii All entries below a leading entry are zeros.

And a matrix is in **reduced row echelon form** (rref) if it is in echelon form and the following additional properties hold

- iv The leading entry of each row is 1 (called a *leading* 1), and
- v each leading 1 is the only nonzero entry in its column.

Example (finding ref's and rref's)

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 0 & 3 & 2 \end{bmatrix}$$

Using elementary row operations, an ref for this matrix is

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

$\frac{1}{2}R_3 \to R_3$ $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The rref for this matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Theorem

Theorem: Each matrix is row equivalent to exactly one reduced row echelon matrix.

This says that the rref of a matrix is unique, and it allows the following unambiguous definition:

Definition: A **pivot position** in a matrix A is a location that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** of a matrix A is a column of A that contains a pivot position.

Identify the pivot position and columns given...

$$A \qquad\qquad\qquad \text{rref of } A$$

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

○ - these are the pivot positions

The pivot columns are columns 1, 2, and 4.

Complete Row Reduction isn't needed to find Pivots

Find the pivot positions and pivot columns of the matrix

$$\begin{bmatrix} 1 & 1 & 4 \\ -2 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$$

○ - pivot positions

Columns 1 and 2 are the pivot columns.

This matrix has an ref and rref

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{respectively.}$$

The locations of the leading 1s are already apparent, in the ref.

Row Reduction Algorithm

To obtain an echelon form, we work from left to right beginning with the top row working downward.

$$\begin{bmatrix} 0 & 3 & -6 & 4 & 6 \\ 3 & -7 & 8 & 8 & -5 \\ 3 & -9 & 12 & 6 & -9 \end{bmatrix}$$

Step 1: Unless the left most column is all zeros, the left most column is a pivot column. The top position is a pivot position.
Step 2: Get a nonzero entry in the top left position by row swapping if needed.

Do $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Step 3: Use row operations to get zeros in all entries below the pivot.

$-R_1 + R_2 \to R_2$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Step 4: Ignore the row with a pivot, all rows above it, the pivot column, and all columns to its left, and repeat steps 1-3 for each pivot position.

Let's do $\frac{1}{2}R_2 \to R_2$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Clear the 3 in row 3 column 2

$$\begin{matrix} 0 & -3 & 6 & -3 & -6 \\ 0 & 3 & -6 & 4 & 6 \end{matrix}$$

Row Reduction Algorithm

$-3R_2 + R_3 \to R_3$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This is an ref.

Row Reduction Algorithm

To obtain a reduced row echelon form: Starting with the right most pivot and working up and to the left, use row operations to get a zero in each position above a pivot. Scale to make each pivot a 1.

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\frac{1}{3}R_1 \to R_1$

$$\begin{bmatrix} 1 & -3 & 4 & 2 & -3 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$-R_3 + R_2 \to R_2$

$-2R_3 + R_1 \to R_1$

Row Reduction Algorithm

$$\begin{bmatrix} 1 & -3 & 4 & 0 & -3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Clear -3 in row $\underline{1}$ column 2

$3R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This is the rref

Echelon Form & Solving a System

Remark: The row operations used to get an rref correspond to an **equivalent** system!

Consider the reduced echelon matrix, and describe the solution set for the associated system of equations (the one who'd have this as its augmented matrix).

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 \qquad\qquad &= 3 \\ x_3 \qquad - 2x_5 &= 4 \\ x_4 \qquad &= -9 \\ 0 &= 0 \end{aligned}$$

$x_1 = 3 - x_2$

$x_3 = 4 + 2x_5$, x_2 and x_5 can be any real numbers.

$x_4 = -9$

Basic & Free Variables

Suppose a system has m equations and n variables, $x_1, x_2, \ldots, x_n$. The first n columns of the augmented matrix correspond to the n variables.

- If the i^{th} column is a pivot column, then x_i is called a **basic variable**.
- If the i^{th} column is NOT a pivot column, then x_i is called a **free variable**.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system would have 4 equations in 5 variables. The basic variables are x_1 x_3 and x_4. The free variables are x_2 and x_5.