## January 23 Math 2306 sec. 51 Spring 2023

### **Section 3: Separation of Variables**

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x} + 1.$$

$$y = \int \frac{dy}{dx} dx$$

$$= \int (4e^{2x} + 1) dx$$

$$y = Ze^{2x} + x + C$$

$$a = 1-parameter family of solutions.$$

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### Separable Equations

**Definition:** The first order equation y' = f(x, y) is said to be **separable** if the right side has the form

$$f(x,y)=g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Determine which (if any) of the following are separable.

(a) 
$$\frac{dy}{dx} = x^3y$$

It is separable

where  $3x = x^3$  here  $3x = 4$ 

(b) 
$$\frac{dy}{dx} = 2x + y$$
 Not separable

(c) 
$$\frac{dy}{dx} = \sin(xy^2)$$

(d) 
$$\frac{dy}{dt} - te^{t-y} = 0$$
  $\Rightarrow \frac{dy}{dt} = te^{t-y} = te^{t-y}$   
 $g(t) = te^{t}$ ,  $h(y) = e^{t}$   
16 is Separable

## Solving Separable Equations

Recall that from  $\frac{dy}{dx} = g(x)$ , we can integrate both sides

$$\int \frac{dy}{dx} dx = \int g(x) dx.$$

Recall that if y is a differentiable function of x, then the differential  $dy = \frac{dy}{dx}dx$ . We have

$$\int dy = \int g(x) dx \implies y = G(x) + C$$

where G is an antiderivative of g.

#### We'll use this observation!



# Solving Separable Equations

Let's assume that it's safe to divide by h(y) and let's set p(y) = 1/h(y). We solve (usually find an implicit solution) by **separating the** variables.

$$\frac{dy}{dx} = g(x)h(y)$$

$$\frac{1}{h(y)} \frac{dy}{dx} = g(x)$$

$$\frac{dy}{dx} = g(x)$$

$$\int b(\lambda) d\lambda = \int \delta(\kappa) d\kappa$$

'Pan = Gixi + C

P is an antiderivative of p and G is on antiderivative of g.

we have a 1-parameter fairly of solutions defined implicitly.

An IVP<sup>1</sup> Find an explicit solution.

$$\frac{dQ}{dt} = -2(Q-70), \quad Q(0) = 180$$

This is separable, 
$$g(t) = -2$$
, and  $h(Q) = Q - 70$ 

$$\frac{1}{Q-70} \frac{dQ}{dt} = -2 \Rightarrow \frac{1}{Q-70} \frac{dQ}{dt} dt = -2 dt$$

$$\int \frac{1}{Q-70} dQ = \int -z dt$$

$$\int \ln |Q-70| = -zt + C$$



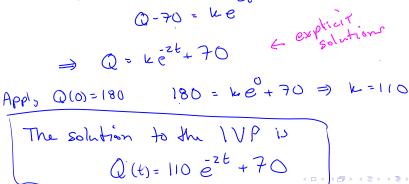
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<sup>&</sup>lt;sup>1</sup>Recall IVP stands for *initial value problem*.

Lets 
$$s_{1}$$
 for  $Q$ 

$$e^{\ln |Q-70|} = e^{-zt+C} \Rightarrow |Q-70| - e^{zt} \cdot e^{C}$$

$$|Q-70| = |Q-70| = |Q-70|$$
or  $Q_{1}$   $k=\pm e^{C}$   $z_{2}$ 



### Solve the IVP

Apply 400=0

$$\frac{dy}{dx} = 4x\sqrt{y}, \quad y(0) = 0$$

$$2\sqrt{y}^{1/2} = 2x^2 + C$$

$$\frac{1}{\sqrt{y}} \frac{\partial y}{\partial x} = 4x \implies \sqrt{y}^{-1/2} \frac{\partial y}{\partial x} dx = 4x dx$$

$$\int \sqrt{y}^{1/2} dy = \int 4x dx$$

$$2\sqrt{y}^{1/2} = 2x^2 + C$$

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C = 0

The solution to the IVP is given implicitly by  $2y^{1/2} = 2x^2$ 

Solving for y explicitly  $(y'^2)^2 = (\chi^2)^2$   $y = \chi^4$ This is an explicit solution to the

IMP.

### Missed Solution

We made an assumption about being able to divide by h(y) when solving  $\frac{dy}{dx} = g(x)h(y)$ . This may cause us to not find valid solutions.

The IVP 
$$\frac{dy}{dx} = 4x\sqrt{y}$$
,  $y(0) = 0$  has two distinct solutions  $y = x^4$ , and  $y(x) = 0$ .

The second solution **CANNOT** be found by separation of variables. Why?



Missed Solutions 
$$\frac{dy}{dx} = g(x)h(y)$$
.

**Theorem:** If the number c is a zero of the function h, i.e. h(c) = 0, then the constant function y(x) = c is a solution to the differential equation  $\frac{dy}{dx} = g(x)h(y)$ .

**Remark:** Such a constant solution may or may not be recovered by separation of variables. We can always look for such solutions in addition to separation of variables.