January 23 Math 2306 sec. 51 Spring 2023

Section 3: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x} + 1.$$

$$y = \int \frac{dy}{dx}\, dx$$

$$= \int (4e^{2x} + 1)\, dx$$

$$y = 2e^{2x} + x + C$$

a 1-parameter family of solutions.

Separable Equations

Definition: The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Determine which (if any) of the following are separable.

(a) $\dfrac{dy}{dx} = x^3 y$

It is separable

w/ $g(x) = x^3$, $h(y) = y$

(b) $\dfrac{dy}{dx} = 2x + y$ Not separable

(c) $\dfrac{dy}{dx} = \sin(xy^2)$

Not separable

(d) $\dfrac{dy}{dt} - te^{t-y} = 0$ $\implies$ $\dfrac{dy}{dt} = te^{t-y} = te^{t}e^{-y}$

$g(t) = te^{t}$, $h(y) = e^{-y}$

It is separable

Solving Separable Equations

Recall that from $\frac{dy}{dx} = g(x)$, we can integrate both sides

$$\int \frac{dy}{dx}\, dx = \int g(x)\, dx.$$

Recall that if y is a differentiable function of x, then the differential $dy = \dfrac{dy}{dx} dx$. We have

$$\int dy = \int g(x)\, dx \quad \Longrightarrow \quad y = G(x) + C$$

where G is an antiderivative of g.

We'll use this observation!

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables.**

$$\frac{dy}{dx} = g(x)h(y)$$ Divide by $h(y)$

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$ multiply by dx

$$p(y)\underbrace{\frac{dy}{dx}\,dx}_{dy} = g(x)\,dx$$

$$p(y)\,dy = g(x)\,dx$$ Integrate

$$\int p(y)\, dy = \int g(x)\, dx$$

$$P(y) = G(x) + C$$

P is an antiderivative of p and G is an antiderivative of g.

we have a 1-parameter family of solutions defined implicitly.

An IVP[1] Find an explicit solution.

$$\frac{dQ}{dt} = -2(Q-70), \quad Q(0) = 180$$

This is separable, $g(t) = -2$, and $h(Q) = Q - 70$

Divide by $h(Q)$

$$\frac{1}{Q-70}\frac{dQ}{dt} = -2 \Rightarrow \frac{1}{Q-70}\underbrace{\frac{dQ}{dt}\,dt}_{dQ} = -2\,dt$$

$$\int \frac{1}{Q-70}\,dQ = \int -2\,dt$$

$$\ln|Q-70| = -2t + C$$

1 parameter family implicit

[1] Recall IVP stands for *initial value problem*.

Let's solve for Q

$$e^{\ln|Q-70|} = e^{-2t+C} \Rightarrow |Q-70| = e^{-2t}\cdot e^{C}$$

Let $k = e^{C}$

$$|Q-70| = ke^{-2t}$$

or Let $k = \pm e^{C}$

$$Q-70 = ke^{-2t}$$

$$\Rightarrow Q = ke^{-2t} + 70 \quad \leftarrow \text{explicit solutions}$$

Apply $Q(0)=180$ $\quad 180 = ke^{0} + 70 \Rightarrow k = 110$

The solution to the IVP is

$$\boxed{Q(t) = 110e^{-2t} + 70}$$

Solve the IVP

$$\frac{dy}{dx} = 4x\sqrt{y}, \quad y(0) = 0$$

$$2y^{1/2} = 2x^2 + C$$

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} = 4x \quad \Rightarrow \quad y^{-1/2}\frac{dy}{dx}\,dx = 4x\,dx$$

$$\int y^{-1/2}\,dy = \int 4x\,dx$$

$$2y^{1/2} = 2x^2 + C$$

Apply $y(0) = 0$ $\quad 2\sqrt{0} = 2(0)^2 + C$

$C = 0$

$\boxed{\text{The solution to the IVP is given implicitly by } 2y^{1/2} = 2x^2}$

Solving for y explicitly

$$(y^{1/2})^2 = (x^2)^2$$

$$y = x^4$$

This is an explicit solution to the IVP.

Missed Solution

We made an assumption about being able to divide by $h(y)$ when solving $\frac{dy}{dx} = g(x)h(y)$. This may cause us to not find valid solutions.

The IVP $\frac{dy}{dx} = 4x\sqrt{y}, \quad y(0) = 0$ has two distinct solutions

$$y = x^4, \quad \text{and} \quad y(x) = 0.$$

The second solution **CANNOT** be found by separation of variables.
Why?

Missed Solutions $\frac{dy}{dx} = g(x)h(y)$.

Theorem: If the number c is a zero of the function h, i.e. $h(c) = 0$, then the constant function $y(x) = c$ is a solution to the differential equation $\frac{dy}{dx} = g(x)h(y)$.

Remark: Such a constant solution may or may not be recovered by separation of variables. We can always look for such solutions in addition to separation of variables.