January 23 Math 2306 sec. 52 Spring 2023

Section 3: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x} + 1.$$

$y = \int \frac{dy}{dx}\, dx = \int (4e^{2x} + 1)\, dx$

$= 2e^{2x} + x + C$

$y = 2e^{2x} + x + C$ is a 1-parameter family of solutions to the ODE.

Separable Equations

Definition: The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Determine which (if any) of the following are separable.

(a) $\dfrac{dy}{dx} = x^3 y$

It is separable

$g(x) = x^3$, $h(y) = y$

(b) $\dfrac{dy}{dx} = 2x + y$

It isn't separable

(c) $\dfrac{dy}{dx} = \sin(xy^2)$

This is not separable

(d) $\dfrac{dy}{dt} - te^{t-y} = 0$

This is separable

$$\frac{dy}{dt} = t\,e^{t-y} = te^{t}e^{-y}$$

$$g(t) = te^{t}, \quad h(y) = e^{-y}$$

Solving Separable Equations

Recall that from $\frac{dy}{dx} = g(x)$, we can integrate both sides

$$\int \frac{dy}{dx}\, dx = \int g(x)\, dx.$$

Recall that if y is a differentiable function of x, then the differential $dy = \dfrac{dy}{dx} dx$. We have

$$\int dy = \int g(x)\, dx \quad \Longrightarrow \quad y = G(x) + C$$

where G is an antiderivative of g.

We'll use this observation!

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables.**

$$\frac{dy}{dx} = g(x)h(y)$$

Divide by $h(y)$

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

mult. by dx

$$p(y)\underbrace{\frac{dy}{dx}\,dx}_{dy} = g(x)\,dx$$

$$p(y)\,dy = g(x)\,dx$$

Integrate both sides

$$\int p(y)\,dy = \int g(x)\,dx$$

$$P(y) = G(x) + C$$

where $P'(y) = p(y)$, $G'(x) = g(x)$

We get a 1-parameter family of solutions given implicitly.

An IVP[1]

Find an explicit solution.

$$\frac{dQ}{dt} = -2(Q-70), \quad Q(0) = 180$$

The ODE is separable w/ $g(t) = -2$, $h(Q) = Q-70$

$$\frac{1}{Q-70}\frac{dQ}{dt} = -2 \Rightarrow \frac{1}{Q-70}\underbrace{\frac{dQ}{dt}dt}_{dQ} = -2dt$$

$$\frac{1}{Q-70}dQ = -2dt \Rightarrow \int \frac{1}{Q-70}dQ = \int -2dt$$

$$\ln|Q-70| = -2t + C$$

1 parameter family of implicit solutions

[1] Recall IVP stands for *initial value problem*.

Let's solve for Q.

$$e^{\ln|Q-70|} = e^{-2t+C} \implies |Q-70| = e^{-2t} \cdot e^{C}$$

Let $k = e^{C}$ or $k = -e^{C}$

$$Q - 70 = k e^{-2t} \implies Q = 70 + k e^{-2t}$$

(param. fam. family of explicit solns.)

Apply $Q(0) = 180$

$$180 = 70 + k e^{0} \implies k = 180 - 70 = 110$$

$$\boxed{\text{The solution to the IVP is } Q = 70 + 110 e^{-2t}}$$

Solve the IVP

$$\frac{dy}{dx} = 4x\sqrt{y}, \quad y(0) = 0$$

This is separable w/ $g(x) = 4x$, $h(y) = \sqrt{y}$

$$\frac{1}{\sqrt{y}} \frac{dy}{dx} = 4x \implies \frac{1}{\sqrt{y}} \frac{dy}{dx}\, dx = 4x\, dx$$

$$\int y^{-1/2}\, dy = \int 4x\, dx$$

$$2y^{1/2} = 2x^2 + C \implies 2\sqrt{y} = 2x^2 + C$$

Apply $y(0)=0$ $\quad 2\sqrt{0} = 2(0)^2 + C \implies C = 0$

The solution to the IVP is given implicitly by

$$2\sqrt{y} = 2x^2$$

We can solve for y.

$$(\sqrt{y})^2 = (x^2)^2$$

$$y = x^4$$

This is an explicit solution to the IVP

Missed Solution

We made an assumption about being able to divide by $h(y)$ when solving $\dfrac{dy}{dx} = g(x)h(y)$. This may cause us to not find valid solutions.

The IVP $\dfrac{dy}{dx} = 4x\sqrt{y}, \quad y(0) = 0$ has two distinct solutions

$$y = x^4, \quad \text{and} \quad y(x) = 0.$$

The second solution **CANNOT** be found by separation of variables.
Why?

We divided by $\sqrt{y}$, $\frac{1}{\sqrt{y}}$ isn't defined if $y=0$.

Missed Solutions $\frac{dy}{dx} = g(x)h(y)$.

Theorem: If the number c is a zero of the function h, i.e. $h(c) = 0$, then the constant function $y(x) = c$ is a solution to the differential equation $\frac{dy}{dx} = g(x)h(y)$.

Remark: Such a constant solution may or may not be recovered by separation of variables. We can always look for such solutions in addition to separation of variables.