

Section 1.2: Row Reduction and Echelon Forms

- ▶ We defined row echelon forms (ref) and reduced row echelon forms (rref).
- ▶ We defined pivot positions and pivot columns.
- ▶ And, we've seen the row reduction algorithm.

Basic & Free Variables

Suppose a system has m equations and n variables, x_1, x_2, \dots, x_n . The first n columns of the augmented matrix correspond to the n variables.

- ▶ If the i^{th} column is a pivot column, then x_i is called a **basic variable**.
- ▶ If the i^{th} column is NOT a pivot column, then x_i is called a **free variable**.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system would have 4 equations in 5 variables. The basic variables are x_1 , x_3 and x_4 . The free variables are x_2 and x_5 .

Basic & Free Variables

When expressing the solution set of a consistent system with infinitely many solutions, we will **always** express basic variables in terms of free variables, and never vice versa.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution to this system will be expressed as

$$x_1 = 3 - x_2$$

$$x_3 = 4 + 2x_5$$

$$x_4 = -9$$

x_2, x_5 are free

x_2 & x_5 can be any real number

Consistent versus Inconsistent Systems

Consider each rref and the corresponding system. Note whether the system is consistent.

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{rclcl} x_1 & + & 0x_2 & + & 2x_3 & = & 3 \\ 0x_1 & + & 1x_2 & + & 1x_3 & = & 0 \\ 0x_1 & + & 0x_2 & + & 0x_3 & = & 1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix}, \quad \begin{array}{rclcl} x_1 & + & 0x_2 & + & 0x_3 & = & 0 \\ 0x_1 & + & 1x_2 & + & 0x_3 & = & 4 \\ 0x_1 & + & 0x_2 & + & 1x_3 & = & -3 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{array}{rclcl} x_1 & + & 2x_2 & + & 0x_3 & = & 0 \\ 0x_1 & + & 0x_2 & + & x_3 & = & 4 \\ 0x_1 & + & 0x_2 & + & 0x_3 & = & 0 \end{array}$$

An Existence and Uniqueness Theorem

Theorem: A linear system is consistent if and only if the right most column of the augmented matrix is **NOT** a pivot column. That is, if and only if each echelon form **DOES NOT** have a row of the form

$$[0 \ 0 \ \cdots \ 0 \ b], \quad \text{for some nonzero } b.$$

If a linear system is consistent, then it has

- (i) exactly one solution if there are **no free variables**, or
- (ii) infinitely many solutions if there is at least one free variable.

Section 1.3: Vector Equations

Definition: A matrix that consists of one column is called a **column vector** or simply a **vector**.

When we give a vector a name (i.e. use a variable to denote a vector), the convention

- ▶ in typesetting is to use bold face

u and **x**

- ▶ in handwriting is to place a little arrow over the variable

\vec{u} and \vec{x}

\mathbb{R}^2

The set of vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with x_1 and x_2 any real numbers is denoted by

 \mathbb{R}^2

(read "R two"). It's the set of all real ordered pairs.

Geometry

Each vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ corresponds to a point in the Cartesian plane. We can equate them with ordered pairs written in the traditional format

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1, x_2).$$

This is **not to be confused with a row matrix**.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq [x_1 \ x_2]$$

We can identify vectors with points or with directed line segments emanating from the origin (little arrows).

Geometry

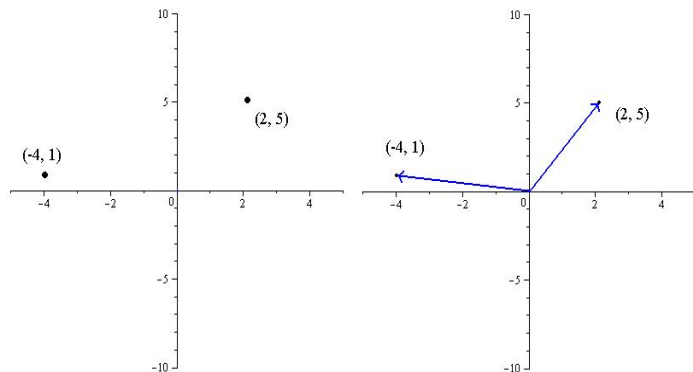


Figure: Vectors characterized as points, and vectors characterized as directed line segments.

$$\begin{bmatrix} -4 \\ 1 \end{bmatrix} = (-4, 1), \quad \begin{bmatrix} 2 \\ 5 \end{bmatrix} = (2, 5)$$

Vector Equality

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and c be a scalar*.

Vector Equivalence: Equality of vectors is defined by

$$\mathbf{u} = \mathbf{v} \quad \text{if and only if} \quad u_1 = v_1 \quad \text{and} \quad u_2 = v_2.$$

*A **scalar** is an element of the set from which u_1 and u_2 come. For our purposes, a scalar is a *real* number.

Algebraic Operations

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and c be a scalar.

Scalar Multiplication: The scalar multiple of \mathbf{u}

$$c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}.$$

Vector Addition: The sum of vectors \mathbf{u} and \mathbf{v}

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

Examples

$$\text{Let } \mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad \text{and } \mathbf{w} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$$

Evaluate

$$(a) \quad -2\mathbf{u} = -2 \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -2(4) \\ -2(-2) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

Examples

$$\text{Let } \mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad \text{and } \mathbf{w} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$$

Evaluate

$$(b) \quad -2\mathbf{u} + 3\mathbf{v} \quad \text{we know } -2\vec{u} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

$$3\vec{v} = 3 \begin{bmatrix} -1 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 21 \end{bmatrix}$$

$$\text{so } -2\vec{u} + 3\vec{v} = \begin{bmatrix} -8 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 21 \end{bmatrix} = \begin{bmatrix} -11 \\ 25 \end{bmatrix}$$

Examples

$$\text{Let } \mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad \text{and } \mathbf{w} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$$

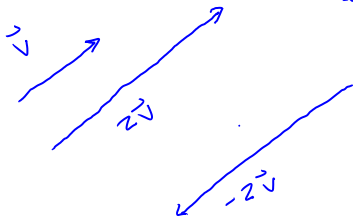
(c) Is it true that $\mathbf{w} = -\frac{3}{4}\mathbf{u}$?

$$\text{Note } -\frac{3}{4}\vec{u} = \begin{bmatrix} -\frac{3}{4}(4) \\ -\frac{3}{4}(-2) \end{bmatrix} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$$

Yes $\vec{w} = -\frac{3}{4}\vec{u}$ since they have the same first and second entry.

Geometry of Algebra with Vectors

Scalar Multiplication: stretches or compresses a vector but can only change direction by an angle of 0 (if $c > 0$) or π (if $c < 0$). We'll see that $0\mathbf{u} = (0, 0)$ for any vector \mathbf{u} .

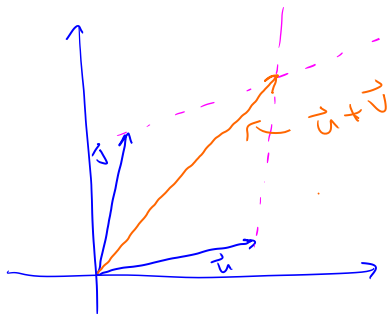


$2\vec{v}$ is parallel and twice as long.

$-2\vec{v}$ is parallel, twice as long and in the direction 180° from \vec{v} .

Geometry of Algebra with Vectors

Vector Addition: The sum $\mathbf{u} + \mathbf{v}$ of two vectors (each different from $(0, 0)$) is the the fourth vertex of a parallelogram whose other three vertices are (u_1, u_2) , (v_1, v_2) , and $(0, 0)$.



Geometry of Algebra with Vectors

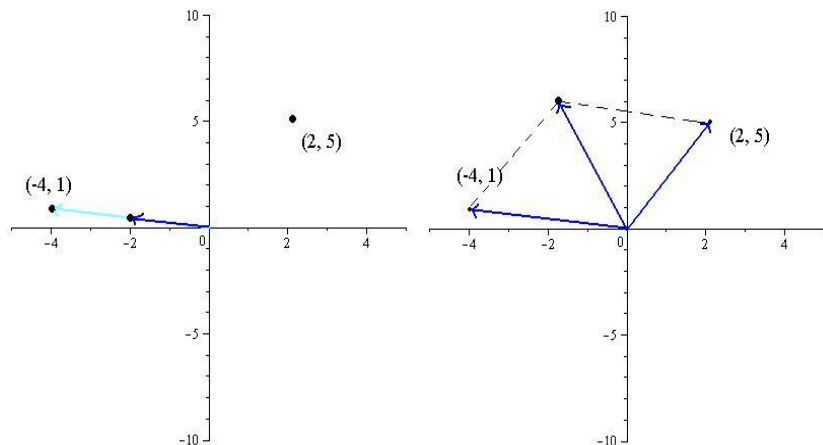


Figure: Left: $\frac{1}{2}(-4, 1) = (-2, 1/2)$. Right: $(-4, 1) + (2, 5) = (-2, 6)$