# January 24 Math 3260 sec. 51 Spring 2024

## Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

**Definition**

Let $A$ be an $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_n$ (each in $\mathbb{R}^m$), and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then the product of $A$ and $\mathbf{x}$, denoted by

$$A\mathbf{x}$$

is the linear combination of the columns of $A$ whose weights are the corresponding entries in $\mathbf{x}$. That is

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

**Remark:** Note that based on the definition of scalar multiplication and vector addition, the product is a vector in $\mathbb{R}^m$.

# Example: Find the product $A\mathbf{x}$.

$m = 2, \quad n = 3$

$\vec{x}$ is in $\mathbb{R}^3$.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$A\vec{x} = 2\vec{a}_1 + 1\vec{a}_2 + (-1)\vec{a}_3$$

$$= 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

# Example: Find the product $A\mathbf{x}$.

$$A = \begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$m=3, \quad n=2$

$\vec{x}$ is in $\mathbb{R}^2$

$\vec{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$

$$A\vec{x} = -3\vec{a}_1 + 2\vec{a}_2 = -3\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

# Example

Is the product $A\mathbf{x}$ defined if $A = \begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$?

No, A has 2 columns and $\vec{x}$ has 3 entries.

# Linear Systems, Vector Equations, & Matrix Equations

Write the linear system as a vector equation and then as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$.

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & x_3 & = & 2 \\ x_1 & + & x_2 & + & & = & -1 \end{array}$$

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

vector equation.

By the definition of the product $A\vec{x}$,

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

# Theorem

**Theorem**

If $A$ is the $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_n$, and $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}]. \leftarrow [A \ \vec{b}]$$

# Corollary

**Corollary**

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of $A$.

**Remark**

In other words, if $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]$, then the corresponding linear system, $A\mathbf{x} = \mathbf{b}$, is consistent if and only if $\mathbf{b}$ is in $\mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

# Example

Characterize the set of all vectors $\mathbf{b} = (b_1, b_2, b_3)$ such that $A\mathbf{x} = \mathbf{b}$ has a solution where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}.$$

The equation $A\vec{x} = \vec{b}$ is equivalent to the system having augmented matrix $[A \ \vec{b}]$.

We can use row reduction.

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \quad \begin{matrix} 4R_1 + R_2 \to R_2 \\ 3R_1 + R_3 \to R_3 \end{matrix} \quad \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \end{bmatrix} \quad -2R_2 + R_3 \to R_3$$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \\ 0 & 0 & 0 & b_2 + 4b_1 - 2(b_3 + 3b_1) \end{bmatrix}$$

The system is only consistent if the last column is <u>not</u> a pivot column.

This requires the entry in the bottom right to be zero.

$$-2b_1 + b_2 - 2b_3 = 0$$

This is a linear system w/ augmented matrix $\begin{bmatrix} -2 & 1 & -2 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -\frac{1}{2} & 1 & 0 \end{bmatrix}$

This gives $b_1 = \frac{1}{2}b_2 - b_3$.

The system is consistent if

$$\vec{b} = \begin{bmatrix} \frac{1}{2}b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix}, \text{ for } b_2, b_3 \text{ in } \mathbb{R}.$$