January 24 Math 3260 sec. 52 Spring 2022

Section 1.2: Row Reduction and Echelon Forms

- We defined row echelon forms (ref) and reduced row echelon forms (rref).
- We defined pivot positions and pivot columns.
- And, we've seen the row reduction algorithm.

Basic & Free Variables

Suppose a system has m equations and n variables, $x_1, x_2, \ldots, x_n$. The first n columns of the augmented matrix correspond to the n variables.

- If the i^{th} column is a pivot column, then x_i is called a **basic variable**.
- If the i^{th} column is NOT a pivot column, then x_i is called a **free variable**.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system would have 4 equations in 5 variables. The basic variables are x_1 x_3 and x_4. The free variables are x_2 and x_5.

Basic & Free Variables

When expressing the solution set of a consistent system with infinitely many solutions, we will **always** express basic variables in terms of free variables, and never vice versa.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution to this system will be expressed as

$$\begin{aligned} x_1 &= 3 - x_2 \\ x_3 &= 4 + 2x_5 \\ x_4 &= -9 \\ x_2, x_5 & \quad \text{are free} \end{aligned}$$

x_2 and x_5 are any real numbers

Consistent versus Inconsistent Systems

Consider each rref and the corresponding system. Note whether the system is consistent.

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{array}{rcrcrcr} x_1 & + & 0x_2 & + & 2x_3 & = & 3 \\ 0x_1 & + & 1x_2 & + & 1x_3 & = & 0 \\ 0x_1 & + & 0x_2 & + & 0x_3 & = & 1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix}, \qquad \begin{array}{rcrcrcr} x_1 & + & 0x_2 & + & 0x_3 & = & 0 \\ 0x_1 & + & 1x_2 & + & 0x_3 & = & 4 \\ 0x_1 & + & 0x_2 & + & 1x_3 & = & -3 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 0x_3 & = & 0 \\ 0x_1 & + & 0x_2 & + & x_3 & = & 4 \\ 0x_1 & + & 0x_2 & + & 0x_3 & = & 0 \end{array}$$

An Existence and Uniqueness Theorem

Theorem: A linear system is consistent if and only if the right most column of the augmented matrix is **NOT** a pivot column. That is, if and only if each echelon form **DOES NOT** have a row of the form

$$[0\ 0\ \cdots\ 0\ b], \quad \text{for some nonzero } b.$$

If a linear system is consistent, then it has

(i) exactly one solution if there are **no free variables**, or
(ii) infinitely many solutions if there is at least one free variable.

Section 1.3: Vector Equations

Definition: A matrix that consists of one column is called a **column vector** or simply a **vector**.

When we give a vector a name (i.e. use a variable to denote a vector), the convention

- in typesetting is to use bold face

$$\mathbf{u} \quad \text{and} \quad \mathbf{x}$$

- in handwriting is to place a little arrow over the variable

$$\vec{u} \quad \text{and} \quad \vec{x}$$

$\mathbb{R}^2$

The set of vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

x_1, x_2 are components or entries

with x_1 and x_2 any real numbers is denoted by

$$\mathbb{R}^2$$

(read ”R two”). It’s the set of all real ordered pairs.

Geometry

Each vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ corresponds to a point in the Cartesian plane. We can equate them with ordered pairs written in the traditional format

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1, x_2).$$

This is **not to be confused with a row matrix.**

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq [x_1 \;\; x_2]$$

We can identify vectors with points or with directed line segments emanating from the origin (little arrows).

Geometry

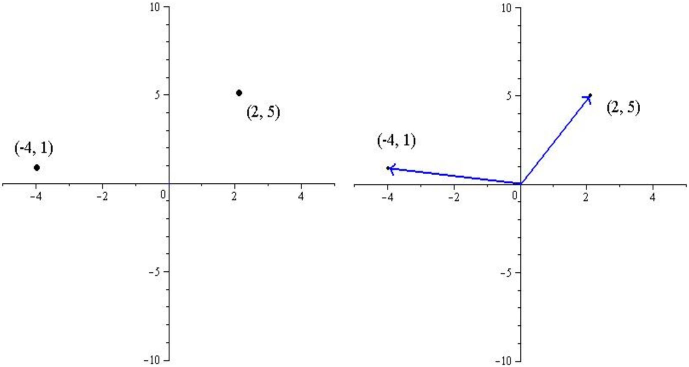


Figure: Vectors characterized as points, and vectors characterized as directed line segments.

$$\begin{bmatrix} -4 \\ 1 \end{bmatrix} = (-4, 1), \quad \begin{bmatrix} 2 \\ 5 \end{bmatrix} = (2, 5)$$

Vector Equality

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and c be a scalar*.

Vector Equivalence: Equality of vectors is defined by

$$\mathbf{u} = \mathbf{v} \quad \text{if and only if} \quad u_1 = v_1 \quad \text{and} \quad u_2 = v_2.$$

*A **scalar** is an element of the set from which u_1 and u_2 come. For our purposes, a scalar is a *real* number.

Algebraic Operations

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and c be a scalar.

Scalar Multiplication: The scalar multiple of **u**

$$c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}.$$

Vector Addition: The sum of vectors **u** and **v**

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

Examples

$$\text{Let} \quad \mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$$

Evaluate

(a) $-2\mathbf{u} = -2 \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -2(4) \\ -2(-2) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$

Examples

$$\text{Let} \quad \mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$$

Evaluate

(b) $-2\mathbf{u}+3\mathbf{v}$

we know $-2\vec{u} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$

$3\vec{v} = 3\begin{bmatrix} -1 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 21 \end{bmatrix}$, so

$$-2\vec{u} + 3\vec{v} = \begin{bmatrix} -8 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 21 \end{bmatrix} = \begin{bmatrix} -11 \\ 25 \end{bmatrix}$$

Examples

$$\text{Let} \quad \mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$$

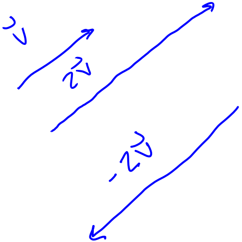
(c) Is it true that $\mathbf{w} = -\frac{3}{4}\mathbf{u}$?

$$\frac{-3}{4}\vec{u} = \begin{bmatrix} \frac{-3}{4}(4) \\ \frac{-3}{4}(-2) \end{bmatrix} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$$

yes $\vec{w} = \frac{-3}{4}\vec{u}$ because they have 1st and 2nd component in common.

Geometry of Algebra with Vectors

Scalar Multiplication: stretches or compresses a vector but can only change direction by an angle of 0 (if $c > 0$) or π (if $c < 0$). We'll see that $0\mathbf{u} = (0, 0)$ for any vector $\mathbf{u}$.

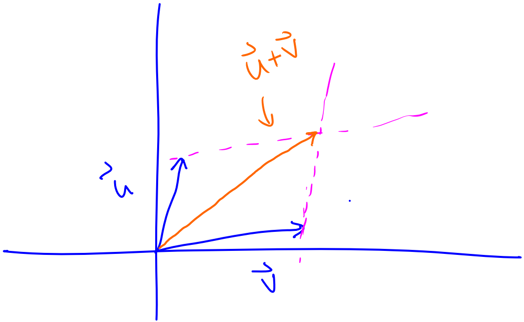


$2\vec{v}$ is parallel to $\vec{v}$ and twice the length.

$-2\vec{v}$ would be parallel but flipped $180°$

Geometry of Algebra with Vectors

Vector Addition: The sum $\mathbf{u} + \mathbf{v}$ of two vectors (each different from $(0, 0)$) is the the fourth vertex of a parallelogram whose other three vertices are (u_1, u_2), (v_1, v_2), and $(0, 0)$.



Geometry of Algebra with Vectors

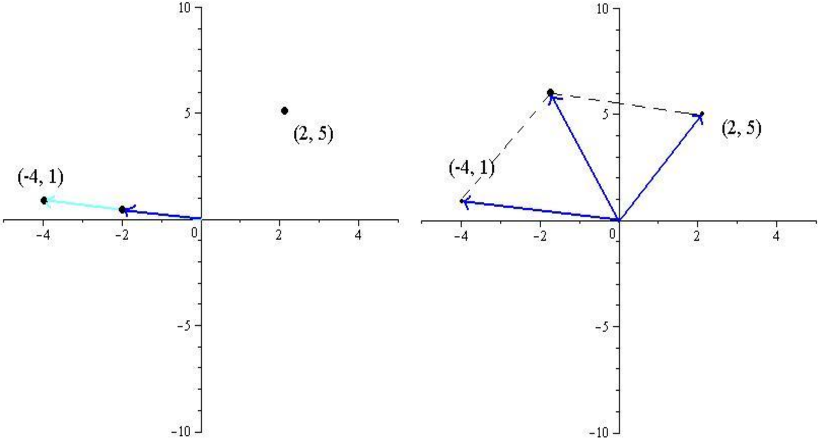


Figure: Left: $\frac{1}{2}(-4,1) = (-2,1/2)$. Right: $(-4,1)+(2,5) = (-2,6)$