January 24 Math 3260 sec. 52 Spring 2024

Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

Definition

Let A be an $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_n$ (each in $\mathbb{R}^m$), and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then the product of A and $\mathbf{x}$, denoted by

$$A\mathbf{x}$$

is the linear combination of the columns of A whose weights are the corresponding entries in $\mathbf{x}$. That is

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

Remark: Note that based on the definition of scalar multiplication and vector addition, the product is a vector in $\mathbb{R}^m$.

Example: Find the product $A\mathbf{x}$.

$m=2, \quad n=3$

$$A = \begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$\vec{x}$ is in $\mathbb{R}^3$

$\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$$

$$= 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 0 \\ -1 \end{bmatrix} + (-1)\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

Example: Find the product $A\mathbf{x}$.

$$A = \begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$m = 3, \; n = 2$

$\vec{x}$ is in $\mathbb{R}^2$

$$\vec{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$A\vec{x} = -3 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

Example

Is the product $A\mathbf{x}$ defined if $A = \begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$?

A has two columns, $\vec{x}$ has 3 components. $A\vec{x}$ is not defined.

Linear Systems, Vector Equations, & Matrix Equations

Write the linear system as a vector equation and then as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$.

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & x_3 & = & 2 \\ x_1 & + & x_2 & + & & = & -1 \end{array}$$

Vector eqn.

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Matrix eqn.

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Theorem

Theorem

If A is the $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_n$, and $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}]. \leftarrow [A \ \vec{b}]$$

Corollary

Corollary

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Remark

In other words, if $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]$, then the corresponding linear system, $A\mathbf{x} = \mathbf{b}$, is consistent if and only if $\mathbf{b}$ is in $\mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

Example

Characterize the set of all vectors $\mathbf{b} = (b_1, b_2, b_3)$ such that $A\mathbf{x} = \mathbf{b}$ has a solution where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}.$$

We can consider the linear system having augmented matrix $[A \ \vec{b}]$.

$$[A \ \vec{b}] = \begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}$$

Do row ops to determine if the last column is a pivot column.

$4R_1 + R_2 \to R_2$

$3R_1 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$
$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & b_3+3b_1 \\ 0 & 14 & 10 & b_2+4b_1 \end{bmatrix}$$

$-2R_2+R_3 \Rightarrow R_3$
$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & b_3+3b_1 \\ 0 & 0 & 0 & b_2+4b_1-2(b_3+3b_1) \end{bmatrix}$$

The last column is not a pivot column, the system is consistent, if

$$b_2+4b_1-2b_3-6b_1=0$$

$$-2b_1+b_2-2b_3=0$$

This is a system whose augmented matrix

is $\begin{bmatrix} -2 & 1 & -2 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -\frac{1}{2} & 1 & 0 \end{bmatrix}$

The solutions are given by

$$b_1 = \frac{1}{2} b_2 - b_3 ,$$

b_2, b_3 are free

$A\vec{x} = \vec{b}$ is solvable if

$$\vec{b} = \begin{bmatrix} \frac{1}{2} b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix}$$ where b_2, b_3 are any real number.