January 25 Math 2306 sec. 51 Spring 2023

Section 3: Separation of Variables

We defined the first order equation as being **separable** if it has the form

$$\frac{dy}{dx} = g(x)h(y).$$

To solve a separable equation, we separate the variables. That is, we convert the ODE to

$$\int \frac{dy}{h(y)} = \int g(x)\,dx$$

and integrate to obtain a one-parameter family of solutions (usually defined implicitly).

Solutions Defined by Integrals

The Fundamental Theorem of Calculus tells us that: If g and $\frac{dy}{dx}$ are continuous on an interval $[x_0, b)$ and x is in this interval, then

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = y(x) - y(x_0).$$

Theorem: If g is continuous on some interval containing x_0, then the function

$$y = y_0 + \int_{x_0}^{x} g(t)\,dt$$

is a solution of the initial value problem

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$

Example

Express the solution of the IVP in terms of an integral.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

$$\boxed{y = y_0 + \int_{x_0}^{x} g(t)\, dt}$$

Here, $g(t) = \sin(t^2)$

$x_0 = \sqrt{\pi}$ and $y_0 = 1$

The solution to the IVP is

$$y = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\, dt$$

Let's verify:

Does $y(\sqrt{\pi}) = 1$? yes!

$$y(\sqrt{\pi}) = 1 + \int_{\sqrt{\pi}}^{\sqrt{\pi}} \sin(t^2)\,dt = 1 + 0 = 1$$

Is $\frac{dy}{dx} = \sin(x^2)$? yes

$$\frac{dy}{dx} = \frac{d}{dx}\left(1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt\right)$$

$$= \frac{d}{dx}(1) + \frac{d}{dx}\int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt$$

$$= 0 + \sin(x^2)$$ i.e. $\frac{dy}{dx} = \sin(x^2)$

Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$P(x) = \frac{a_0(x)}{a_1(x)}$

$$\frac{dy}{dx} + P(x)y = f(x).$$

$f(x) = \frac{g(x)}{a_1(x)}$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the equation
$$\frac{dy}{dx} + P(x)y = 0$$
(called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!

Motivating Example

$$x^2\frac{dy}{dx}+2xy = e^x$$

Let's assume that $x > 0$.

Note that the left side is the derivative of the product x^2y.

Note that $\frac{d}{dx}(x^2y) = x^2\frac{dy}{dx} + 2xy$

The ODE can be written as

$$\frac{d}{dx}(x^2y) = e^x$$

Integrate w/ respect to x

$$\int \frac{d}{dx}(x^2 y)\,dx = \int e^x\,dx$$

$$x^2 y = e^x + C$$

The solutions are

$$y = \frac{e^x + C}{x^2}$$

A 1-parameter family of solutions