January 26 Math 3260 sec. 51 Spring 2022

Section 1.3: Vector Equations

We defined a vector (or column vector) as a matrix consisting of a single column.

The set $\mathbb{R}^2$ is the set of all real ordered pairs $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where x_1 and x_2 are real numbers. We equate them in the traditional way with points in the Cartesian plane.

The components of the vector, i.e. the entries in the vector as a matrix, are referred to as **scalars**.

Algebraic Operations

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and c be a scalar.

Vector Equivalence: Equality of vectors is defined by

$$\mathbf{u} = \mathbf{v} \quad \text{if and only if} \quad u_1 = v_1 \quad \text{and} \quad u_2 = v_2.$$

Scalar Multiplication: The scalar multiple of **u**

$$c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}.$$

Vector Addition: The sum of vectors **u** and **v**

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

Geometry of Algebra with Vectors

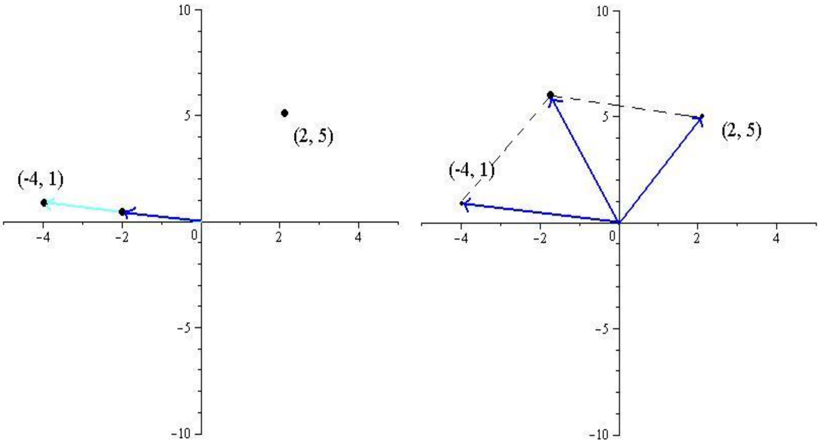


Figure: Left: $\frac{1}{2}(-4,1) = (-2,1/2)$. Right: $(-4,1) + (2,5) = (-2,6)$

Vectors in $\mathbb{R}^3$ ("R three")

A vector in $\mathbb{R}^3$ is a 3×1 column matrix. For example

$$\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \text{or} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Similar to vectors in $\mathbb{R}^2$, vectors in $\mathbb{R}^3$ are ordered triples.

$$\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = (1, 3, -1).$$

Vectors in $\mathbb{R}^n$ (R n)

A vector in $\mathbb{R}^n$ for $n \geq 2$ is a $n \times 1$ column matrix. These are ordered n-tuples. For example

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The Zero Vector: is the vector whose entries are all zeros. It will be denoted by $\mathbf{0}$ or $\vec{0}$ and is not to be confused with the scalar 0.

Scalar multiplication and vector addition will be defined component-wise in $\mathbb{R}^n$

Algebraic Properties on $\mathbb{R}^n$

For every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$ and scalars c and d[1]

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = d(c\mathbf{u}) = (cd)\mathbf{u}$

(viii) $1\mathbf{u} = \mathbf{u}$

[1] The term $-\mathbf{u}$ denotes $(-1)\mathbf{u}$.

Definition: Linear Combination

A linear combination of vectors $\mathbf{v}_1, \ldots \mathbf{v}_p$ in $\mathbb{R}^n$ is a vector $\mathbf{y}$ of the form

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

where the scalars $c_1, \ldots, c_p$ are often called weights.

For example, suppose we have two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Some linear combinations include

$$3\mathbf{v}_1, \quad -2\mathbf{v}_1 + 4\mathbf{v}_2, \quad \frac{1}{3}\mathbf{v}_2 + \sqrt{2}\mathbf{v}_1, \quad \text{and} \quad \mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2.$$

Example

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$. Determine if $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

$\vec{b}$ is a linear combination of $\vec{a}_1$ and $\vec{a}_2$ if there exists scalars c_1 and c_2 such that $c_1\vec{a}_1 + c_2\vec{a}_2 = \vec{b}$. This would give the equation

$$c_1 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ -2c_1 \\ -c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 \\ 0 \\ 2c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + 3c_2 \\ -2c_1 \\ -c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$$

Vector equality is defined componentwise, so this is equivalent to the linear system

$$\begin{aligned} c_1 + 3c_2 &= -2 \\ -2c_1 \quad &= -2 \\ -c_1 + 2c_2 &= -3 \end{aligned}$$

We could use an augmented matrix

$$\begin{bmatrix} 1 & 3 & -2 \\ -2 & 0 & -2 \\ -1 & 2 & -3 \end{bmatrix} \xrightarrow[\text{TI 92}]{\text{rref}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

↑ not a pivot column

The system is consistent, From the rref

$c_1 = 1$ and $c_2 = -1$.

So $\vec{b} = 1\,\vec{a}_1 - 1\,\vec{a}_2$.

Hence $\vec{b}$ is a linear combination of $\vec{a}_1$ and $\vec{a}_2$.

Some Convenient Notation

Letting $\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$, and in general $\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$, for $j = 1, ..., n$, we can denote the $m \times n$ matrix whose columns are these vectors by

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Note that each vector $\mathbf{a}_j$ is a vector in $\mathbb{R}^m$.

Vector and Matrix Equations

The vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}. \tag{1}$$

In particular, $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if the linear system whose augmented matrix is given in (1) is consistent.

Definition of **Span**

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{R}^n$. The set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \text{Span}(S).$$

It is called the **subset of $\mathbb{R}^n$ spanned by (a.k.a. generated by)** the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

To say that a vector $\mathbf{b}$ is in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ means that there exists a set of scalars $c_1, \ldots, c_p$ such that $\mathbf{b}$ can be written as

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p.$$

Span

If $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, then $\mathbf{b} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$. From the previous result, we know this is equivalent to saying that the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

has a solution. This is in turn the same thing as saying the linear system with augmented matrix $[\mathbf{v}_1 \ \cdots \ \mathbf{v}_p \ \mathbf{b}]$ is consistent.

Examples

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{a}_2 = \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}$.

(a) Determine if $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$.

This is equivalent to determining if the system of equations w/ augmented matrix $[\vec{a}_1\ \vec{a}_2\ \vec{b}]$ is consistent.

$$[\vec{a}_1\ \vec{a}_2\ \vec{b}] = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & 2 \\ 2 & -2 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

↖ pivot column

The system is inconsistent

Hence $\vec{b}$ is not in Span $\{\vec{a}_1, \vec{a}_2\}$.

(b) Determine if $\mathbf{b} = \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$.

$$[\vec{a}_1 \ \vec{a}_2 \ \vec{b}] = \begin{bmatrix} 1 & -1 & 5 \\ 1 & 4 & -5 \\ 2 & -2 & 10 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Yes, $\vec{b}$ is in span$\{\vec{a}_1, \vec{a}_2\}$. In fact

$$\vec{b} = 3\vec{a}_1 - 2\vec{a}_2$$

Another Example

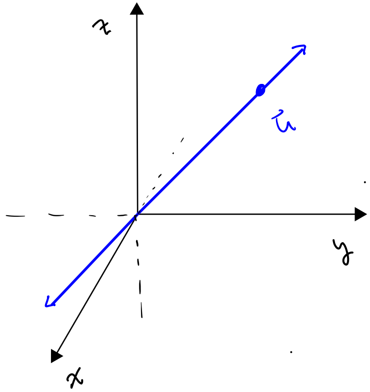
Give a geometric description of the subset of $\mathbb{R}^2$ given by
$\text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$.

This contains all vectors of the form $c\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}c\\0\end{bmatrix}$ for all possible values of c.

Note $\begin{bmatrix}c\\0\end{bmatrix}=(c,0)$ This is the line $y=0$ a.k.a the x-axis.

Span{**u**} in $\mathbb{R}^3$

If **u** is any nonzero vector in $\mathbb{R}^3$, then Span{**u**} is a line through the origin parallel to **u**.



line through the points (0,0,0) and (u_1, u_2, u_3)

Span$\{\mathbf{u}, \mathbf{v}\}$ in $\mathbb{R}^3$

If **u** and **v** are nonzero, and nonparallel vectors in $\mathbb{R}^3$, then Span$\{\mathbf{u}, \mathbf{v}\}$ is a plane containing the origin parallel to both vectors.

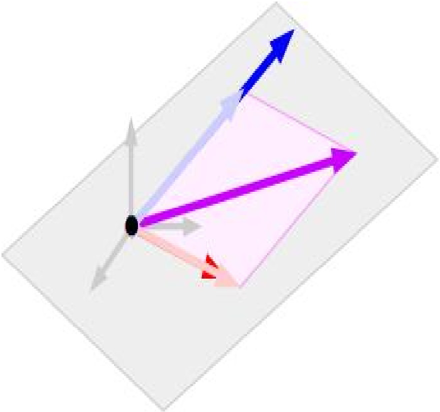


Figure: The red and blue vectors are **u** and **v**. The plane is the collection of all possible linear combinations. (A purple representative is shown.)