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Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

Theorem

If A is the $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$, and $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}].$$

We saw that the matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the subset of $\mathbb{R}^m$ spanned by the columns of A.

Example

Characterize the set of all vectors $\mathbf{b} = (b_1, b_2, b_3)$ such that $A\mathbf{x} = \mathbf{b}$ has a solution where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}.$$

We set up the augmented matrix $[A\ \mathbf{b}]$ and did row reduction to conclude that the equation is consistent provided

$$b_1 = \frac{1}{2}b_2 - b_3, \quad \text{i.e.,} \quad \mathbf{b} = \begin{bmatrix} \frac{1}{2}b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix}$$

where b_2 and b_3 are any real numbers.

Now, lets express the vectors $\mathbf{b}$ in terms of a span.

Example Continued

Find a set S of appropriate fixed vectors in $\mathbb{R}^3$ so that we can say that $A\mathbf{x} = \mathbf{b}$ is consistent provided $\mathbf{b}$ is in $\text{Span}(S)$.

$$\mathbf{b} = \begin{bmatrix} \frac{1}{2}b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}b_2 \\ b_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -b_3 \\ 0 \\ b_3 \end{bmatrix}$$

$$= b_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

This is a linear combo of

$\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

We can say that $A\vec{x} = \vec{b}$ is solvable if $\vec{b}$ is in $\text{Span}\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Theorem (first in a string of equivalency theorems)

Theorem

Let A be an $m \times n$ matrix. Then the following are logically equivalent (i.e. they are either all true or are all false).

(a) For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

Remark: That last statement, (d), is about *coefficient* matrix A. It's not about an augmented matrix $[A \quad \mathbf{b}]$.

A Special Product

Definition

Consider two vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ in $\mathbb{R}^n$. The **dot product** of $\mathbf{x}$ and $\mathbf{y}$, denoted

$$\mathbf{x} \cdot \mathbf{y},$$

is defined by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Remark: Note that the dot product of two vectors is a scalar. This is an example of an *inner product*. It's sometimes called a *scalar product*.

Computing $A\mathbf{x}$

The dot product can be used as an alternative way of computing a product $A\mathbf{x}$. If A is an $m \times n$ matrix and $\mathbf{x}$ is a vector in $\mathbb{R}^n$, then the i^{th} component of the product $A\mathbf{x}$ is the dot product of the i^{th} **row** of A and the vector $\mathbf{x}$.

For example

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} (1)(2) + (0)(1) + (-3)(-1) \\ (-2)(2) + (-1)(1) + (4)(-1) \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

Evaluate

Use the dot product approach to compute each product.

$$\begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3(2)+2(4) \\ -3(-1)+2(1) \\ -3(0)+2(3) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

3×2 in $\mathbb{R}^2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1x_1+0x_2+0x_3 \\ 0x_1+1x_2+0x_3 \\ 0x_1+0x_2+1x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

3×3 $\mathbb{R}^3$

Identity Matrix

Definition: Identity Matrix

We'll call an $n \times n$ matrix with 1's on the diagonal and 0's everywhere else—i.e. one that looks like

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

the $n \times n$ **identity** matrix and denote it by I_n. (We'll drop the subscript if it's obvious from the context.)

This matrix has the property that for each $\mathbf{x}$ in $\mathbb{R}^n$

$$I_n\mathbf{x} = \mathbf{x}.$$

Properties of the Matrix Product

Theorem

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is any scalar, then

(a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, and

(b) $A(c\mathbf{u}) = cA\mathbf{u}$.

Remark: These two properties are a pretty **big deal**! These are the two properties that constitute being **Linear** (as in *Linear Algebra*).

Section 1.5: Solution Sets of Linear Systems

Definition

A linear system is said to be **homogeneous** if it can be written in the form

$$A\mathbf{x} = \mathbf{0}$$

for some $m \times n$ matrix A and where $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Theorem

Theorem: A homogeneous system $A\mathbf{x} = \mathbf{0}$ always has at least one solution, $\mathbf{x} = \mathbf{0}$, called the **trivial solution**.

Homogeneous Linear Systems

We know that the homogeneous system

$$A\mathbf{x} = \mathbf{0}$$

is always consistent because $\mathbf{x} = \mathbf{0}$ (in $\mathbb{R}^n$) is necessarily a solution.

The interesting question is:

Does $A\mathbf{x} = \mathbf{0}$ have any **nontrivial** *solutions?*

Theorem

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the system has at least one free variable.

Homogeneous Linear Systems

Example: Determine if the homogeneous system has a nontrivial solution. Describe the solution set.

(a) $$\begin{array}{rcrcl} 2x_1 & + & x_2 & = & 0 \\ x_1 & - & 3x_2 & = & 0 \end{array}$$

In matrix format

$$\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can use an augmented matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

There are no free variables. The system has only the trivial solution,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\text{(b)}\quad \begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

In matrix format

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using an augmented matrix

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the rref, x_3 is free. Therefore are nontrivial solutions. The solutions satisfy

$$x_1 = \frac{4}{3}x_3$$
$$x_2 = 0$$
$$x_3 \text{ is free}$$

this is called parametric form

We can express this as a vector,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

this is called parametric vector form.

(c) $x_1 - 2x_2 + 5x_3 = 0$

We can use an augmented matrix

$\begin{bmatrix} 1 & -2 & 5 & 0 \end{bmatrix}$ which is already in rref.

The solutions are given by

$x_1 = 2x_2 - 5x_3$

x_2, x_3 are free.

There are nontrivial solutions. In vector form,

$$\vec{x} = \begin{bmatrix} 2x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}.$$

The solutions are in $\mathrm{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}$.