January 27 Math 2306 sec. 51 Spring 2023

Section 4: First Order Equations: Linear

We consider a first order linear ODE in standard form

$$\frac{dy}{dx} + P(x)y = f(x),$$

and assume that P and f are continuous on the interval of definition.

The general solution will have the form

$$y = y_c + y_p$$

where y_c is called the **complementary solution** and y_p is called a **particular solution**.

Motivating Example

$$x^2\frac{dy}{dx}+2xy=e^x \qquad \Longrightarrow \qquad \frac{d}{dx}\left(x^2y\right)=e^x$$

We solved this ODE by recognizing that the left side collapses as $\frac{d}{dx}\left(x^2y\right)$. We got the one-parameter family of solutions

$$y=\frac{e^x+C}{x^2}.$$

This can be expressed as

$$y=\frac{e^x}{x^2}+\frac{C}{x^2}.$$

The complementary solution $y_c=\frac{C}{x^2}$, and the particular solution $y_p=\frac{e^x}{x^2}$.

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

want $\frac{d}{dx}(\text{something}) = \text{something else}$

we find a function $\mu(x)$ such that when we multiply the equation by μ the left side becomes $\frac{d}{dx}(\mu y)$. Suppose μ exists and $\mu(x) > 0$. Multiply by μ

$$\mu \frac{dy}{dx} + \mu P(x) y = \mu f(x)$$

Note $\frac{d}{dx}(\mu y) = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y$

Set this equal to the left side above.

$$\mu \frac{dy}{dx} + \frac{d\mu}{dx} y = \mu \frac{dy}{dx} + \mu P(x) y$$

This requires $\frac{d\mu}{dx} y = \mu P(x) y$

Divide by y.

$$\frac{d\mu}{dx} = \mu P(x)$$

This is separable.

Separate variables

$$\frac{1}{\mu}\frac{d\mu}{dx} = P(x)$$

$$\frac{1}{\mu}\,d\mu = P(x)\,dx$$

$$\ln \mu = \int P(x)\,dx$$

$$\Rightarrow \quad \mu = e^{\int P(x)\,dx}$$

This is called an integrating factor.

Going back to the ODE

$$\mu \frac{dy}{dx} + \mu P(x) y = \mu f(x)$$

for $\mu = e^{\int P(x)\,dx}$ we get

$$\frac{d}{dx}(\mu y) = \mu f(x)$$

$$\int \frac{d}{dx}(\mu y)\,dx = \int \mu f(x)\,dx$$

$$\mu y = \int \mu f(x)\,dx$$

$$\boxed{y = \frac{1}{\mu} \int \mu f(x)\,dx}$$

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\, dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\, dx = e^{-\int P(x)\, dx}\left(\int e^{\int P(x)\, dx} f(x)\, dx + C\right)$$

Solve the IVP

$$\frac{dy}{dx} + P(x)y = f(x)$$

$$x\frac{dy}{dx} - y = 2x^2, \; x > 0 \quad y(1) = 5$$

Standard form

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{2x^2}{x} = 2x$$

$$P(x) = \frac{-1}{x}. \quad \text{Build } \mu = e^{\int P(x)\,dx}$$

$$\int P(x)\,dx = \int \frac{-1}{x}\,dx = -\ln x$$

we can take any antiderivative of P

$$\mu = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

Multiply by μ

$$x^{-1}\left(\frac{dy}{dx} - \frac{1}{x}y\right) = x^{-1}(2x)$$

Collapse the left

$$\frac{d}{dx}\left(x^{-1}y\right) = 2$$

$$\int \frac{d}{dx}\left(x^{-1}y\right)dx = \int 2\,dx$$

$$x^{-1}y = 2x + C$$

$$y = \frac{2x+C}{x^{-1}} = 2x^2 + Cx$$

The solutions to the ODE are

$y = 2x^2 + Cx$

$* \quad y_p = 2x^2$

$y_c = Cx$

Apply $y(1) = 5$

$y(1) = 2(1^2) + C(1) = 5 \Rightarrow 2 + C = 5$

$C = 3$

$$\boxed{\text{The solution to the IVP is } \quad y = 2x^2 + 3x}$$

Verify

Just for giggles, lets verify that our solution $y = 2x^2 + 3x$ really does solve the differential equation we started with

$$x\frac{dy}{dx} - y = 2x^2.$$

$y = 2x^2 + 3x$, $\quad y' = 4x + 3$ sub into the ODE

$$x(4x+3) - (2x^2 + 3x) \overset{?}{=} 2x^2$$

$$4x^2 + 3x - 2x^2 - 3x \overset{?}{=} 2x^2$$

$$2x^2 \overset{?}{=} 2x^2$$

yes this is true.