January 27 Math 2306 sec. 52 Spring 2023

Section 4: First Order Equations: Linear

We consider a first order linear ODE in standard form

$$\frac{dy}{dx} + P(x)y = f(x),$$

and assume that P and f are continuous on the interval of definition.

The general solution will have the form

$$y = y_c + y_p$$

January 25, 2023

1/26

where y_c is called the **complementary solution** and y_p is called a **particular solution**.

Motivating Example

$$x^{2}\frac{dy}{dx}+2xy=e^{x}$$
 $\frac{d}{dx}(x^{2}y)=e^{x}$

We solved this ODE by recognizing that the left side collapses as $\frac{d}{dx}(x^2y)$. We got the one-parameter family of solutions

$$y=\frac{e^{x}+C}{x^{2}}.$$

This can be expressed as

$$y=\frac{e^x}{x^2}+\frac{C}{x^2}.$$

January 25, 2023

2/26

The complementary solution $y_c = \frac{C}{x^2}$, and the particular solution $y_p = \frac{e^x}{x^2}$.

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

We want to write this as

$$\frac{d}{dx} (something) = something else
Use'll find a function $\mu(x)$ such that when
we multiply the OE by μ , the left side
becomes $\frac{d}{dx}(\mu y)$, we'll assume μ exists
and that $\mu(x) > 0$. Multiply by, $\mu$$$

$$r \frac{dy}{dx} + \mu P(x) \cdot y = \mu f(x)$$
Node $\frac{d}{dx} (\mu y) = \mu \frac{dy}{dx} + \frac{d}{dx} \cdot y$
we want this to equal the left side.

$$\mu \frac{dy}{dx} + \frac{d}{dx} \cdot y = \mu \frac{dy}{dx} + \mu P(x) \cdot y$$
This will require

$$\frac{d\mu}{dx} \cdot y = \mu P(x) \cdot y$$
Divide by y . So μ satisfier

$$\frac{d\mu}{dx} = \mu P(x)$$

January 25, 2023 4/26

This is separable. Solve $\frac{1}{\mu} \frac{d\mu}{dx} = P(x)$ I dy = Pusdx $\int \frac{1}{m} d\mu = \int P(x) dx$ Jup = JPW Jx JPXIJX JPE E This is an integrating factor. イロト イ団ト イヨト イヨト 二日

For this p

$$r \frac{dy}{dx} + pP(x) g = pf(x)$$

 $\frac{d}{dx}(pg) = pf(x)$
 $\int \frac{d}{dx}(pg) dx = \int pf(x) dx$
 $pg = \int pf(x) dx$
 $y = p \int pf(x) dx$

▲□▶ < @▶ < ≧▶ < ≧▶ ≧ シ ≧ のへで January 25, 2023 7/26

General Solution of First Order Linear ODE

- Put the equation in standard form y' + P(x)y = f(x), and correctly identify the function P(x).
- Obtain the integrating factor $\mu(x) = \exp(\int P(x) dx)$.
- Multiply both sides of the equation (in standard form) by the integrating factor µ. The left hand side will always collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)} \int \mu(x) f(x) \, dx = e^{-\int P(x) \, dx} \left(\int e^{\int P(x) \, dx} f(x) \, dx + C \right)$$

Solve the IVP

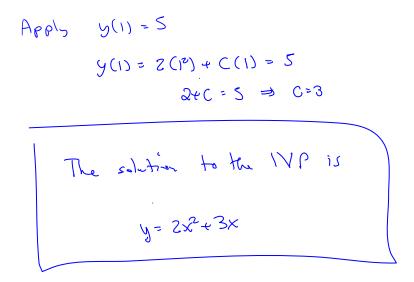
dy + P(x) y = f(x)

9/26

 $x\frac{dy}{dx}-y=2x^2, x>0 \quad y(1)=5$ Standard form: $\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{x}(zx^2) = 2x$ Nore P(x) = = 1 , build pr = e Sp(x) dx Span dx = Stodx = - Inx $\mu = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$ Multiply the ODE by h January 25, 2023

 $\chi^{-1}\left(\frac{dy}{dx}-\frac{1}{x}y\right) = \chi^{-1}(2x)$ Collapse $\frac{d}{dx}(\vec{x}'y) = Z$ $\int \frac{d}{dx} \left(x^{-1} y \right) dx = \int Z dx$ x'y = 2x+C $\implies y = \frac{2x+C}{x^{-1}} = 2x^{2} + Cx$ Solutions to the ODE are y - 2x2 + Cx ◆□▶ ◆□▶ ◆ □ ▶ ◆ □ ▶ ◆ □ ▶

January 25, 2023 10/26



Verify

Just for giggles, lets verify that our solution $y = 2x^2 + 3x$ really does solve the differential equation we started with

$$x\frac{dy}{dx} - y = 2x^{2}.$$

$$y = 2x^{2} + 3x, \quad y' = 4x + 3 \quad \text{sub}:$$

$$x(4x+3) - (2x^{2} + 3x) \stackrel{?}{=} 2x^{2}$$

$$4x^{2} + 3x - 2x^{2} - 3x \stackrel{?}{=} 2x^{2}$$

$$2x^{2} \stackrel{?}{=} 2x^{2}$$

$$y = this is an identity,$$

January 25, 2023 12/26