January 28 Math 3260 sec. 52 Spring 2022

Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

Definition Let A be an $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ (each in $\mathbb{R}^m$), and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then the product of A and $\mathbf{x}$, denoted by

$$A\mathbf{x}$$

is the linear combination of the columns of A whose weights are the corresponding entries in $\mathbf{x}$. That is

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(Note that the result is a vector in $\mathbb{R}^m$!)

Example

Find the product $A\mathbf{x}$. Simplify to the extent possible.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

A is a 2×3 matrix and $\vec{x}$ is a vector in $\mathbb{R}^3$.

The vectors $\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\vec{a}_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, and $\vec{a}_3 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

are in $\mathbb{R}^2$.

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$$

$$= 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 0 \\ -1 \end{bmatrix} + (-1)\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

Example

Find the product $A\mathbf{x}$. Simplify to the extent possible.

$$A = \begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

A is 3×2 and $\vec{x}$ is in $\mathbb{R}^2$.

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2$$

$$= -3\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

Example

Write the linear system as a vector equation and then as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$.

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & x_3 & = & 2 \\ x_1 & + & x_2 & + & & = & -1 \end{array}$$

As a vector equation, this is

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

As a matrix equation, this is

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

The matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Theorem

If A is the $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$, and $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}].$$

This can be written as $[A \ \vec{b}]$

Corollary

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

In other words, the corresponding linear system is consistent if and only if $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

Example

Characterize the set of all vectors $\mathbf{b} = (b_1, b_2, b_3)$ such that $A\mathbf{x} = \mathbf{b}$ has a solution where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}.$$

We can do this by considering the linear system that has augmented matrix $[A \ \vec{b}]$.

$$[A \ \vec{b}] = \begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}$$

We can do row reduction to an ref

$4R_1 + R_2 \to R_2$ and $3R_1 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \end{bmatrix}$$

$-2R_2 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \\ 0 & 0 & 0 & -2b_1 + b_2 - 2b_3 \end{bmatrix}$$

This is an ref

The system with augmented matrix $[A\ \vec{b}]$ is consistent if and only if that last column is <u>not</u> a pivot column.

This requires $-2b_1 + b_2 - 2b_3 = 0$.

So $A\vec{x} = \vec{b}$ is solvable only if

$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ where $-2b_1 + b_2 - 2b_3 = 0$.