January 29 Math 3260 sec. 51 Spring 2024

Section 1.5: Solution Sets of Linear Systems

Definition

A linear system is said to be **homogeneous** if it can be written in the form

$$A\mathbf{x} = \mathbf{0}$$

for some $m \times n$ matrix A and where $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Theorems

Theorem 1: A homogeneous system $A\mathbf{x} = \mathbf{0}$ always has at least one solution, $\mathbf{x} = \mathbf{0}$, called the **trivial solution**.

Theorem 2: The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the system has at least one free variable.

Examples from last time:

We used an augmented matrix to identify solution sets.

(a) $\begin{array}{rcrcl} 2x_1 & + & x_2 & = & 0 \\ x_1 & - & 3x_2 & = & 0 \end{array}$ trivial solution only $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(b) $\begin{array}{rcrcrcl} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$ nontrivial solutions

$$\mathbf{x} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}, \quad x_3 \text{ is free}$$

(c) $x_1 - 2x_2 + 5x_3 = 0$ nontrivial solutions

$$\mathbf{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \text{ are free}$$

Parametric Vector Form of a Solution Set

Example (b) had a solution set consisting of vectors of the form $\mathbf{x} = x_3\mathbf{v}$. Example (c)'s solution set consisted of vectors that look like $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$. Instead of using the variables x_2 and/or x_3 we often substitute **parameters** such as s or t.

Parametric Vector Form of a Solution Set

The forms

$$\mathbf{x} = s\mathbf{u}, \quad \text{or} \quad \mathbf{x} = s\mathbf{u} + t\mathbf{v}$$

$s \in \mathbb{R}$
or
$-\infty < t < \infty$

are called **parametric vector forms.**

Remark: Since these are **linear combinations**, an alternative way to express the solution sets would be

$$\mathrm{Span}\{\mathbf{u}\} \quad \text{or} \quad \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}.$$

Geometry

The **parametric vector form** of the solution set of the system

$$\begin{array}{rcrcrcl} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array} \quad \text{is}$$

$$\mathbf{x} = s\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}, \quad s \in \mathbb{R}.$$

" $\in$ "
in or an element of

This is a line in $\mathbb{R}^3$ through the points $(0,0,0)$ and $\left(\frac{4}{3},0,1\right)$.

Geometry

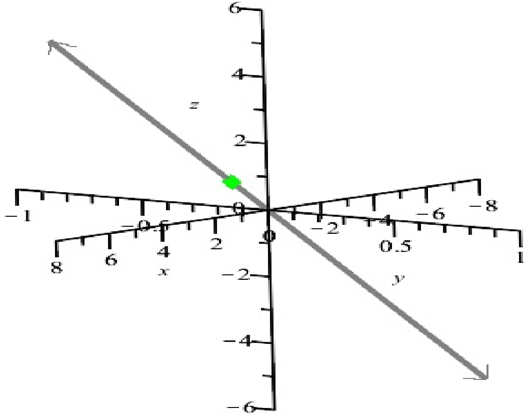

Figure: Plot of the line $\mathbf{x} = s \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$. The point $\left(\frac{4}{3}, 0, 1\right)$ is shown in green.

Geometry

The **parametric vector form** of the solution set of
$x_1 \; - \; 2x_2 \; + \; 5x_3 \; = \; 0$ is

$$\mathbf{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \quad \text{where } s, t \in \mathbb{R}.$$

This is a plane in $\mathbb{R}^3$ that contains the points $(0,0,0)$, $(2,1,0)$, and $(-5,0,1)$.

Geometry

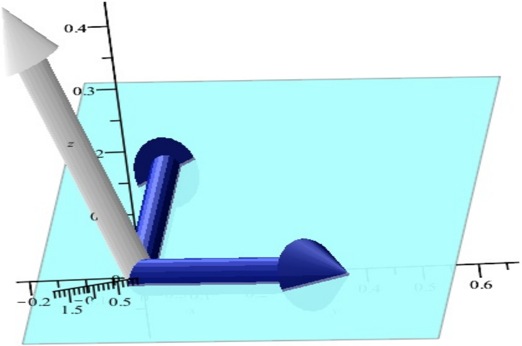

Figure: Plot of the plane $\mathbf{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$. The blue vectors are in the directions of $(2, 1, 0)$ and $(-5, 0, 1)$. (The white vector is perpendicular—a.k.a. *normal*—to the plane.)

Nonhomogeneous Systems

Find all solutions of the nonhomogeneous system of equations

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 7 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & -1 \\ 6x_1 & + & x_2 & - & 8x_3 & = & -4 \end{array}$$

We can use an augmented matrix

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From the rref x_1 and x_2 are basic and x_3 is free.

$x_1 = -1 + \frac{4}{3}x_3$

$x_2 = 2$

x_3 is free

parametric form

We can express this in parametric vector form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3} x_3 \\ 2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}.$$

The solutions are $\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}.$

Geometry $\mathbf{x} = (-1, 2, 0) + t\left(\frac{4}{3}, 0, 1\right)$ in $\mathbb{R}^3$

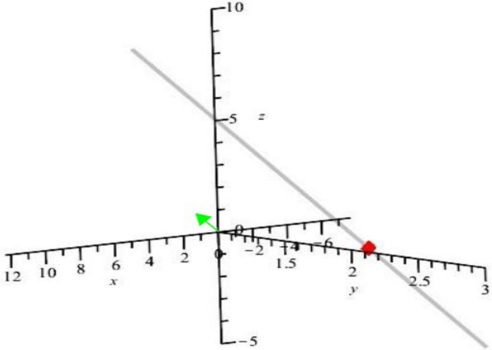

Figure: Plot of the line $\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$. The point $(-1, 2, 0)$ is shown in red, and the vector $\left(\frac{4}{3}, 0, 1\right)$ is shown in green.

Solutions of Nonhomogeneous Systems

Note that the solution in this example has the form

$$\mathbf{x} = \mathbf{p} + t\mathbf{v}$$

with $\mathbf{p}$ and $\mathbf{v}$ fixed vectors and t a varying parameter. Also note that the $t\mathbf{v}$ part is the solution to the previous example with the right hand side all zeros. This is no coincidence!

The vector $\mathbf{p}$ is called a **particular solution**, and $t\mathbf{v}$ is called a solution to the associated homogeneous equation.

General Solution Nonhomogeneous Equation

Theorem

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for a given $\mathbf{b}$. Let $\mathbf{p}$ be a particular solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form

$$\mathbf{x} = \mathbf{p} + \mathbf{v}_h,$$

where $\mathbf{v}_h$ is any solution of the associated homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Remark: We can use a row reduction technique to get all parts of the solution in one process.

Example

Find the solution set of the following system. Express the solution set in parametric vector form.

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & & & + & x_4 & = & 2 \\ 3x_1 & - & 6x_2 & + & x_3 & - & x_4 & = & 7 \end{array}$$

We can use an augmented matrix.

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 2 \\ 3 & -6 & 1 & -1 & 7 \end{bmatrix} \quad -3R_1 + R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -4 & 1 \end{bmatrix}$$

From the rref, x_1 and x_3 are basic. x_2 and x_4 are free.

$x_1 = 2 + 2x_2 - x_4$

x_2 - free

$x_3 = 1 + 4x_4$

x_4 - free

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 + 2x_2 - x_4 \\ x_2 \\ 1 + 4x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

The solutions are

$$\vec{x} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{p}} + \underbrace{t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 4 \\ 1 \end{bmatrix}}_{\vec{v}_h}, \quad s, t \in \mathbb{R},$$

Section 1.7: Linear Independence

We already know that a homogeneous equation $A\mathbf{x} = \mathbf{0}$ can be thought of as an equation in the column vectors of the matrix $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$ as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_n\mathbf{a}_n = \mathbf{0}.$$

And, we know that at least one solution (the trivial one $x_1 = x_2 = \cdots = x_n = 0$) always exists.

Remark: The existence, or not, of a nontrivial solution is a property of the set of vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$.

Definition: Linear Independence

Definition:Linear Independence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

If a set of vectors is not linearly independent, we say that it is **linearly dependent**.

Remark: This definition fully defines **Linear Dependence**. However, we could choose to define linear dependence directly.

Linear Dependence & Independence

Definition: Linear Dependence

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists a set of weights $c_1, c_2, \ldots, c_p$, *at least one of which is nonzero*, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}.$$

Remark: The phrase "*at least one of which is nonzero*" is a reference to a **nontrivial solution**.

Definition: Linear Dependence Relation

An equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}$, with at least one $c_i \neq 0$, is called a **linear dependence relation**.

Theorem on Linear Independence

Theorem:

The columns of a matrix A are linearly **independent** if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Remark: This follows from the definition of linear independence. This connects a homogeneous system $A\mathbf{x} = \mathbf{0}$ with a property of the columns of A as a set of vectors.

Example

(a) Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent or linearly independent.

We can create a matrix, $A = [\vec{v}_1\ \vec{v}_2]$, and look at the homogeneous system $A\vec{x} = \vec{0}$.

Using an augmented matrix

$$[A\ \vec{0}] = \begin{bmatrix} 2 & 1 & 0 \\ 4 & -2 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The rref shows that $A\vec{x} = \vec{0}$ has only the trivial solution.

By the theorem on the last slide, the columns of A are linearly independent.

Hence $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.

Example

(b) Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent or linearly independent.

Note that $\vec{v}_1 + \vec{v}_2 = \vec{v}_3$. Hence

$\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$. This is a

linear dependence relation

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ w/ $c_1 = c_2 = 1$ and $c_3 = -1$

Since at least one coefficient is nonzero, the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent

Example

(c) Determine if the set of vectors is linearly dependent or linearly independent. If dependent, find a linear dependence relation.

$$\left\{ \underbrace{\begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix}}_{\vec{v}_3}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}}_{\vec{v}_4} \right\}$$

Call these $\vec{v}_1$ through $\vec{v}_4$ (in order).

We can use the matrix $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4]$ and the augmented matrix for $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & 3 & 2 & 0 \\ 0 & 1 & 3 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_4 is free, they are lin. dependent.