January 30 Math 2306 sec. 51 Spring 2023

Section 4: First Order Equations: Linear

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

Steady and Transient States

$\frac{dy}{dx} + P(x)\, y = f(x)$

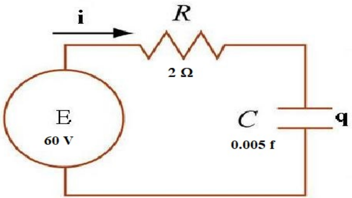

Figure: The charge $q(t)$ on the capacitor in the given curcuit satisfies a first order linear equation.

$$2\frac{dq}{dt} + 200q = 60, \quad q(0) = 0.$$

Standard form: $\frac{dq}{dt} + 100q = 30$

$P(t) = 100$ Build $\mu = e^{\int P(t)dt}$

$$\mu = e^{\int 100\,dt} = e^{100t}$$

$$e^{100t}\left(\frac{dq}{dt} + 100q\right) = e^{100t}(30)$$

$$\frac{d}{dt}\left(e^{100t} q\right) = 30e^{100t}$$

$$\int \frac{d}{dt}\left(e^{100t} q\right) dt = \int 30e^{100t}\,dt$$

$$e^{100t} q = \frac{30}{100} e^{100t} + K$$

$$\int e^{at}\,dt = \frac{1}{a}e^{at} + C$$

$$q = \frac{\frac{3}{10}e^{100t} + K}{e^{100t}}$$

$$q = \frac{3}{10} + k e^{-100t}$$

1-parameter family of solutions

Now, apply $q(0) = 0$

$$q(0) = \frac{3}{10} + k e^{0} = 0 \implies k = \frac{-3}{10}$$

The solution to the IVP is

$$q(t) = \frac{3}{10} - \frac{3}{10} e^{-100t}$$

Steady and Transient States

Note that the solution, the charge, consists of a complementary and a particular solution, $q = q_p + q_c$.

$$q(t) = \frac{3}{10} - \frac{3}{10}e^{-100t}$$

$$q_c(t) = -\frac{3}{10}e^{-100t} \quad \text{and} \quad q_p(t) = \frac{3}{10}$$

Evaluate the limit

$$\lim_{t\to\infty} q_c(t) = \lim_{t\to\infty} \frac{-3}{10} e^{-100t} = 0$$

Steady and Transient States

The complementary solution contains the information given by the initial condition, and for some physical systems like this the complementary solution decays.

Definition: Such a decaying complementary solution is called a **transient state**.

Note that due to this decay, after a while $q(t) \approx q_p(t)$.

Definition: Such a corresponding particular solution is called a **steady state**.

Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

Observation: This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.

Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

We'll define a new dependent variable

$u = y^{1-n}$. Then

$$\frac{du}{dx} = (1-n)\,y^{1-n-1}\frac{dy}{dx} \Rightarrow \frac{du}{dx} = (1-n)\,y^{-n}\frac{dy}{dx}$$

Divide the ODE by y^n.

$$y^{-n}\left(\frac{dy}{dx} + P\cdot y\right) = y^{-n}\left(f(x)\,y^n\right)$$

$$y^{-n}\frac{dy}{dx} + P(x)\, y^{1-n} = f(x)$$

Note that $\frac{1}{1-n}\frac{du}{dx} = y^{-n}\frac{dy}{dx}$

and $u = y^{1-n}$

The ODE for u is

$$\frac{1}{1-n}\frac{du}{dx} + P(x)\, u = f(x)$$

So u solves the linear ODE

$$\frac{du}{dx} + (1-n)P(x)\, u = (1-n)f(x)$$

we can write

$$\frac{du}{dx} + P_1(x)u = f_1(x)$$

where $P_1(x) = (1-n)P(x)$ and $f_1(x) = (1-n)f(x)$

$$u = y^{1-n} \Longrightarrow y = u^{\frac{1}{1-n}}$$

Example

$\frac{dy}{dx} + P(x)y = f(x)y^n$

Solve the initial value problem $y' - y = -e^{2x}y^3$, subject to $y(0) = 1$.

This is Bernoulli w/ $n = 3$, $P(x) = -1$

and $f(x) = -e^{2x}$.

Let $u = y^{1-n} = y^{1-3} = y^{-2}$. u satisfies

$\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x)$, $1-n = -2$

$\frac{du}{dx} + (-2)(-1)u = (-2)(-e^{2x})$

$\frac{du}{dx} + 2u = 2e^{2x}$

$P_1(x) = 2$, $\mu = e^{\int P_1(x)dx} = e^{\int 2dx} = e^{2x}$

$$\frac{d}{dx}\left(e^{2x}u\right) = e^{2x}\left(2e^{2x}\right) = 2e^{4x}$$

$$\int \frac{d}{dx}\left(e^{2x}u\right)dx = \int 2e^{4x}\,dx$$

$$e^{2x}u = \frac{2}{4}e^{4x} + C$$

$$\Longrightarrow \quad u = \frac{\frac{1}{2}e^{4x} + C}{e^{2x}} = \frac{1}{2}e^{2x} + Ce^{-2x}$$

So $\quad u = \frac{1}{2}e^{2x} + Ce^{-2x}$

From $\quad u = y^{-2} \quad \Longrightarrow \quad y = u^{-1/2} = \frac{1}{\sqrt{u}}$

The solutions to the ODE are

$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2x} + Ce^{-2x}}}$$

Apply $y(0) = 1$

$$y(0) = \frac{1}{\sqrt{\frac{1}{2}e^{0} + Ce^{0}}} = 1$$

$$\frac{1}{\sqrt{\frac{1}{2} + C}} = 1$$

$$\sqrt{\frac{1}{2} + C} = 1$$

$$\left(\frac{1}{2} + C\right) = 1^2 = 1$$

$\Rightarrow \quad C = \frac{1}{2}$

$\boxed{\text{The solution to the IVP is} \quad y = \dfrac{1}{\sqrt{\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}}}.}$