# January 30 Math 2306 sec. 52 Spring 2023

## Section 4: First Order Equations: Linear

### General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor $\mu$. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for $y$.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

# Steady and Transient States

$\frac{dy}{dx} + P(x)\, y = f(x)$

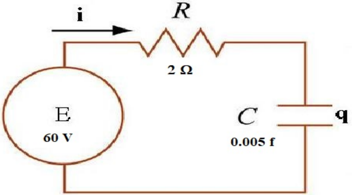


Figure: The charge $q(t)$ on the capacitor in the given curcuit satisfies a first order linear equation.

$$2\frac{dq}{dt} + 200q = 60, \quad q(0) = 0.$$

Standard form: $\frac{dq}{dt} + 100q = 30$

$P(t) = 100$ . Find $\mu = e^{\int p(t)\,dt}$

$$\mu = e^{\int 100\,dt} = e^{100t}$$

Multiply the ODE by $\mu$

$$e^{100t}\left(\frac{dq}{dt} + 100q\right) = e^{100t}(30)$$

$$\frac{d}{dt}\left(e^{100t} q\right) = 30e^{100t}$$

Integrate $\int \frac{d}{dt}\left(e^{100t} q\right) dt = \int 30\, e^{100t}\, dt$

$$e^{100t} q = \frac{30}{100} e^{100t} + k$$

$\int e^{at}\, dt$

$\frac{1}{a} e^{at} + C$

$$q = \frac{\frac{3}{10}e^{100t} + k}{e^{100t}} = \frac{3}{10} + ke^{-100t}$$

$$q(t) = \frac{3}{10} + ke^{-100t}$$

Apply $q(0) = 0$

$$q(0) = \frac{3}{10} + ke^{0} = 0 \implies k = \frac{-3}{10}$$

The solution to the IVP is

$$\boxed{q = \frac{3}{10} - \frac{3}{10}e^{-100t}}$$

# Steady and Transient States

Note that the solution, the charge, consists of a complementary and a particular solution, $q = q_p + q_c$.

$$q(t) = \frac{3}{10} - \frac{3}{10}e^{-100t}$$

$$q_c(t) = -\frac{3}{10}e^{-100t} \quad \text{and} \quad q_p(t) = \frac{3}{10}$$

Evaluate the limit

$$\lim_{t\to\infty} q_c(t) = \lim_{t\to\infty} \frac{-3}{10} e^{-100t} = 0$$

# Steady and Transient States

The complementary solution contains the information given by the initial condition, and for some physical systems like this the complementary solution decays.

**Definition:** Such a decaying complementary solution is called a **transient state**.

Note that due to this decay, after a while $q(t) \approx q_p(t)$.

**Definition:** Such a corresponding particular solution is called a **steady state**.

# Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval $(a, b)$ and $n$ is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

**Observation:** This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.

# Solving the Bernoulli Equation

$$\frac{dy}{dx}+P(x)y=f(x)y^n$$

We'll introduce a new dependent variable called $u$.

Set $u=y^{1-n}$. Let's find $\frac{du}{dx}$

$$\frac{du}{dx}=(1-n)y^{1-n-1}\frac{dy}{dx} \Rightarrow \boxed{\frac{du}{dx}=(1-n)y^{-n}\frac{dy}{dx}}$$

Divide the ODE by $y^n$ (i.e. multiply by $y^{-n}$)

$$y^{-n}\left(\frac{dy}{dx}+P(x)y\right)=y^{-n}\left(f(x)y^n\right)$$

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = f(x)$$

Note $\quad y^{-n}\dfrac{dy}{dx} = \dfrac{1}{1-n}\dfrac{du}{dx}$ and $y^{1-n} = u$

The ODE is

$$\frac{1}{1-n}\frac{du}{dx} + P(x)\,u = f(x)$$

$u$ satisfies the linear ODE

$$\frac{du}{dx} + (1-n)P(x)\,u = (1-n)f(x)$$

we can write

$$\frac{du}{dx} + P_1(x)\,u = f_1(x)$$

where $P_1(x) = (1-n)P(x)$ and

$$f_1(x) = (1-n)\,f(x)$$

$$u = y^{1-n} \implies y = u^{\frac{1}{1-n}}$$

# Example

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Solve the initial value problem $y' - y = -e^{2x}y^3$, subject to $y(0) = 1$.

The ODE is Bernoulli with

$n = 3$, $P(x) = -1$, $f(x) = -e^{2x}$.

Let $u = y^{1-n} = y^{1-3} = y^{-2}$.

Note $1-n = -2$. $u$ solves

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x)$$

$$\rightarrow \quad \frac{du}{dx} + (-2)(-1)u = (-2)(-e^{2x})$$

$$\frac{du}{dx} + 2u = 2e^{2x}$$

1st order linear $P_1(x) = 2$,

so

$$\mu = e^{\int P_1(x)\,dx} = e^{\int 2\,dx} = e^{2x}$$

$$\frac{d}{dx}\left(e^{2x} u\right) = e^{2x}\left(2e^{2x}\right) = 2e^{4x}$$

$$\int \frac{d}{dx}\left(e^{2x} u\right) dx = \int 2e^{4x}\,dx$$

$$e^{2x} u = \frac{2}{4} e^{4x} + C$$

$$u = \frac{\frac{1}{2}e^{4x} + C}{e^{2x}} = \frac{1}{2}e^{2x} + Ce^{-2x}$$

$$u = \frac{1}{2} e^{2x} + C e^{-2x}$$

Solve for $y$. $u = y^{-2} \Rightarrow y = u^{-1/2} = \frac{1}{\sqrt{u}}$

$$\boxed{y = \frac{1}{\sqrt{\frac{1}{2} e^{2x} + C e^{-2x}}}}$$

1 parameter family of soln. to the ODE

Apply $y(0) = 1$.

$$y(0) = \frac{1}{\sqrt{\frac{1}{2} e^{0} + C e^{0}}} = 1$$

$$\Rightarrow \quad \frac{1}{\sqrt{\frac{1}{2}+C}} = 1$$

$$\sqrt{\frac{1}{2}+C} = 1 \qquad C = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{2} + C = 1^2 = 1 \quad \Rightarrow \quad C = \frac{1}{2}$$

The solution to the IVP is

$$\boxed{y = \frac{1}{\sqrt{\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}}}}$$