January 31 Math 3260 sec. 52 Spring 2022

Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

For $m \times n$ matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ and vector $\mathbf{x}$ in $\mathbb{R}^n$, we defined the product

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

So $A\mathbf{x}$ is the vector in $\mathbb{R}^m$ that is the linear combination of the columns of A with the entries of $\mathbf{x}$ as the weights.

Theorem

If A is the $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$, and $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}].$$

We can say that $A\mathbf{x} = \mathbf{b}$ is solvable if and only if $\mathbf{b}$ is a linear combination of the columns of A, i.e. if and only $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

Example

Characterize the set of all vectors $\mathbf{b} = (b_1, b_2, b_3)$ such that $A\mathbf{x} = \mathbf{b}$ has a solution where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}.$$

We set this up by considering the linear system with augmented matrix $[A\,\mathbf{b}]$. We did row reduction to the following ref

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \\ 0 & 0 & 0 & -2b_1 + b_2 - 2b_3 \end{bmatrix}$$

The linear system is consistent if and only if that last column is not a pivot column. That is, the entries of $\mathbf{b}$ must satisfy $-2b_1 + b_2 - 2b_3 = 0$ for $A\mathbf{x} = \mathbf{b}$ to have a solution.

Example

So we know that the vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ must have entries satisfying

$$-2b_1 + b_2 - 2b_3 = 0$$

for the following matrix equation to have a solution.

$$\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

We can express this set (of all such vectors **b**) in terms of a span. That is, we can characterize all such vectors as belonging to $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$ for fixed vectors **u** and **v**.

$-2b_1 + b_2 - 2b_3 = 0$ is a linear system with

augmented matrix $\begin{bmatrix} -2 & 1 & -2 & 0 \end{bmatrix}$

To get an rref, do $-\frac{1}{2}R_1 \to R_1$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 1 & 0 \end{bmatrix}$$ b_1 is basic, b_2, b_3 are free

The solutions are

$b_1 = \frac{1}{2}b_2 - b_3$, b_2 and b_3 are free

So the vector

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}b_2 \\ b_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -b_3 \\ 0 \\ b_3 \end{bmatrix}$$

$$= b_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

This says that $\vec{b}$ is a linear combination of the vectors $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

This can be expressed as $\vec{b}$ is in $\mathrm{Span}\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Theorem (first in a string of equivalency theorems)

Let A be an $m \times n$ matrix. Then the following are logically equivalent (i.e. they are either all true or are all false).

(a) For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

(Note that statement (d) is about the *coefficient* matrix A, not about an augmented matrix $[A \quad \mathbf{b}]$.)

A Special Product

Consider two vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ in $\mathbb{R}^n$. The **dot product**[1] of $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Note that this is a scalar.

[1] This is also called the scalar product or inner product.

Computing $A\mathbf{x}$

We can use a *row-vector* dot product rule. The i^{th} entry is $A\mathbf{x}$ is the sum of products of corresponding entries from row i of A with those of $\mathbf{x}$. That is, it is the dot product of the i^{th} *Row* of A (as a vector in $\mathbb{R}^n$ with the vector $\mathbf{x}$. For example

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1(2)+0(1)+(-3)(-1) \\ -2(2)+(-1)(1)+4(-1) \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

A is 2×3

$\vec{x}$ is in $\mathbb{R}^3$.

$A\vec{x}$ is in $\mathbb{R}^2$

$$\begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(-3)+4(2) \\ -1(-3)+1(2) \\ 0(-3)+3(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

A is 3×2

$\vec{x}$ is in $\mathbb{R}^2$

$A\vec{x}$ is in $\mathbb{R}^3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1x_1+0x_2+0x_3 \\ 0x_1+1x_2+0x_3 \\ 0x_1+0x_2+1x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

this is I_3

Identity Matrix

We'll call an $n \times n$ matrix with 1's on the diagonal and 0's everywhere else—i.e. one that looks like

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

the $n \times n$ **identity** matrix and denote it by I_n. (We'll drop the subscript if it's obvious from the context.)

This matrix has the property that for each $\mathbf{x}$ in $\mathbb{R}^n$

$$I_n \mathbf{x} = \mathbf{x}.$$

Theorem: Properties of the Matrix Product

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is any scalar, then

(a) $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, and

(b) $A(c\mathbf{u}) = cA\mathbf{u}$.

Remark: These two properties are kinda a big deal. The concept of being *Linear*, as in Linear Algebra, is defined in terms of these two properties.

Section 1.5: Solution Sets of Linear Systems

Definition A linear system is said to be **homogeneous** if it can be written in the form

$$A\mathbf{x} = \mathbf{0}$$

for some $m \times n$ matrix A and where $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Theorem: A homogeneous system $A\mathbf{x} = \mathbf{0}$ always has at least one solution $\mathbf{x} = \mathbf{0}$.

The solution $\mathbf{x} = \mathbf{0}$ is called the **trivial solution**. A more interesting question for a homogeneous system is

Does it have a nontrivial solution?

Theorem

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the system has at least one free variable.

Example: Determine if the homogeneous system has a nontrivial solution. Describe the solution set.

(a)
$$\begin{array}{rcrcl} 2x_1 & + & x_2 & = & 0 \\ x_1 & - & 3x_2 & = & 0 \end{array}$$

We can try to solve this by using an augmented matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -3 & 0 \end{bmatrix}$$

Let's do row reduction to rref

Do $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & -3 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

Do $-2R_1 + R_2 \to R_2$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 7 & 0 \end{bmatrix} \leftarrow \text{ref} \qquad \frac{1}{7}R_2 \to R_2$$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad 3R_2 + R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \leftarrow \text{rref}$$

$x_1 = 0$

$x_2 = 0,$

Both x_1 and x_2 are basic.

There are no nontrivial solutions

The solution set is $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

$$
\text{(b)} \quad \begin{array}{rcrcrcl} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}
$$

The augmented matrix is

$$
\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \xrightarrow[\text{TI-92}]{\text{rref}} \begin{array}{c} \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} & & \uparrow & \end{array} \end{array}
$$

non pivot column

There is a free variable hence there are nontrivial solutions.

From the rref

$$
x_1 = \frac{4}{3}x_3
$$
$$
x_2 = 0
$$

x_3 is free

The solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}.$$

The solution set is $\text{Span}\left\{ \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$$\vec{x} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$ ← this is called vector parametric form.

x_3 is any real number.