January 31 Math 3260 sec. 52 Spring 2024

Section 1.7: Linear Independence

Definition:Linear Independence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

Remark: Alternatively, we can say that the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is **linearly dependent** if there exists a set of weights $c_1, c_2, \dots, c_p$, *at least one of which is nonzero*, such that the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}.$$

is satisfied.

Theorem on Linear Independence

Theorem:

The columns of a matrix A are linearly **independent** if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Remark: We can use this result as a tool. Given any set of vectors in $\mathbb{R}^n$, we can always create a matrix from them by just using them as columns.

Example

(b) Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent or linearly independent.

By observation $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$, we can rearrange this to set $\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$. This is a linear dependence relation. Note the coefficients, 1, 1, and -1, are not all zero.

The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

Example

(c) Determine if the set of vectors is linearly dependent or linearly independent. If dependent, find a linear dependence relation.

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Call these $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ in the order given, and let $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4]$.

Consider $A\vec{x} = \vec{0}$.

Setting up an augmented matrix for $A\vec{x} = \vec{0}$,

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & 3 & 2 & 0 \\ 0 & 1 & 3 & 0 & 0 \end{bmatrix} \xrightarrow[\text{using TI 92}]{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \frac{-1}{3} x_4$

$x_2 = -2x_4$

$x_3 = \frac{2}{3} x_4$

x_4 - free

Since x_4 is free $A\vec{x}=\vec{0}$ has nontrivial solutions. The set is linearly dependent.

The vectors satisfy

$$-\frac{1}{3}x_4\vec{v}_1 - 2x_4\vec{v}_2 + \frac{2}{3}x_4\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$$

Setting x_4 to any nonzero number gives a linear dependence relation. For $x_4=-3$, we set

$$\vec{v}_1 + 6\vec{v}_2 - 2\vec{v}_3 - 3\vec{v}_4 = \vec{0}$$

From

$$\begin{bmatrix} 1 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v}_4 = \frac{1}{3}\vec{v}_1 + 2\vec{v}_2 - \frac{2}{3}\vec{v}_3$$

Theorem

Theorem

An indexed set of two or more vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the others in the set.

Example: Let **u** and **v** be any nonzero vectors in $\mathbb{R}^3$. Show that if **w** is any vector in Span$\{\mathbf{u}, \mathbf{v}\}$, then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly **dependent.**

Since $\vec{w}$ is in Span$\{\vec{u}, \vec{v}\}$, there are scalars c_1 and c_2 such that

$$\vec{w} = c_1\vec{u} + c_2\vec{v}.$$

We can rearrange this to get

$$c_1\vec{u} + c_2\vec{v} - \vec{w} = \vec{0}$$

The coefficient on $\vec{w}$ is $-1 \neq 0$.

So this is a linear dependence relation. The set $\{\vec{u}, \vec{v}, \vec{w}\}$ is therefore linearly dependent.

Caveat!

A set may be linearly dependent even if all proper subsets are linearly independent. For example, consider

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Each set $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. (You can easily verify this.)

However,

$$\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1 \quad \text{i.e.} \quad \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0},$$

so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

This means that you can't just consider two vectors at a time.

Two More Theorems

Theorem:

If a set contains more vectors than there are entries in each vector, then the set is linearly **dependent**. That is, if $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is a set of vector in $\mathbb{R}^n$, and $p > n$, then the set is linearly dependent.

For example, if you have 7 vectors, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\}$, and each of these is a vector in $\mathbb{R}^5$, i.e., $\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \\ v_{41} \\ v_{51} \end{bmatrix}$ and so forth, then they must be **linearly dependent** because $7 > 5$.

Two More Theorems

Theorem:

Any set of vectors that contains the zero vector is linearly **dependent.**

Consider the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{0}\}$ in $\mathbb{R}^n$. Note that

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p + 1\mathbf{0} = \mathbf{0}$$

is a **linear dependence relation** because the last coefficient $c_{p+1} = 1$ is nonzero. It doesn't matter what the other vectors are or what the values of p and n are relative to one another!

Examples

Without doing any computations, determine, with justification, whether the given set is linearly dependent or linearly independent.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$

4 vectors in $\mathbb{R}^3$. Since $4 > 3$, they are lin. dependent.

Examples

(b) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -8 \\ 1 \end{bmatrix}, \right\}$

This set contains $\vec{0}$. It's

lin. dependent.