January 8 Math 3260 sec. 52 Spring 2024

A Random Motivational Example

Plop plop, fizz fizz, oh what a relief it is[1].

$$\underbrace{NaHCO_3}_{\text{sodium bicarbonate}} + \underbrace{H_3C_6H_5O_7}_{\text{citric acid}} \longrightarrow \underbrace{Na_3C_6H_5O_7}_{\text{sodium citrate}} + \underbrace{H_2O}_{\text{water}} + \underbrace{CO_2}_{\text{carbon dioxide}}.$$

This is an unbalanced chemical equation that describes effervescence of a commercial antacid medication.

Question: How many molecules of each substance result in a balanced equation?

[1] Sodium bicarbonate and citric acid dissolved in water produces sodium citrate, water, and carbon dioxide.

Motivating Example: Balancing Atoms

$$\underline{x_1}\,\mathrm{NaHCO_3} + \underline{x_2}\,\mathrm{H_3C_6H_5O_7} \longrightarrow \underline{x_3}\,\mathrm{Na_3C_6H_5O_7} + \underline{x_4}\,\mathrm{H_2O} + \underline{x_5}\,\mathrm{CO_2}$$

We can introduce a 4-tuple $\begin{bmatrix} \text{Na} \\ \text{H} \\ \text{C} \\ \text{O} \end{bmatrix}$ and create an equation for the unknowns x_1, x_2, x_3, x_4, and x_5.

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 8 \\ 6 \\ 7 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 5 \\ 6 \\ 7 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

This is an example of the types of equations we want to consider.

We'll work in a variety of settings...

Linear sys.

$$\begin{array}{rcrcrcrcrcl} x_1 & & & - & 3x_3 & & & & & = & 0 \\ x_1 & + & 8x_2 & - & 5x_3 & - & 2x_4 & & & = & 0 \\ x_1 & + & 6x_2 & - & 6x_3 & & & - & x_5 & = & 0 \\ 3x_1 & + & 7x_2 & - & 7x_3 & - & x_4 & - & 2x_5 & = & 0 \end{array}$$

Matrix eqns.

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 1 & 8 & -5 & -2 & 0 \\ 1 & 6 & -6 & 0 & -1 \\ 3 & 7 & -7 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

More Matrices

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 1 & 8 & -5 & -2 & 0 & 0 \\ 1 & 6 & -6 & 0 & -1 & 0 \\ 3 & 7 & -7 & -1 & -2 & 0 \end{bmatrix}$$

Vector eqns.

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 8 \\ 6 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -5 \\ -6 \\ -7 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ 0 \\ -1 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Questions:

- Is there a set of numbers $x_1, \ldots, x_5$ that satisfy all of the equations?
- If there is a set of numbers, is it the only one?
- Are there simple algorithms we can use to answer these questions?

These are some of the questions addressed by Linear Algebra. We'll also consider two main abstractions:

Vector Spaces and **Linear Transformations**.

Section 1.1: Systems of Linear Equations

We begin with a linear (*algebraic*) equation in *n real* variables x_1, x_2, ..., x_n for some positive integer n.

Definition

A **linear equation** in the variables $x_1, \ldots, x_n$ is one that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, \ldots, a_n$ are real (or complex) constants called the *coefficients*, and b is a constant.

In general, the coefficients and the right hand side b are known.

Linear Equation in n Variables

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b.$$

Notice the main structure on the left side. The unknowns/variables $(x_1, \ldots, x_n)$ are

- multipled by numbers (a.k.a. coefficients), and
- added together.

Other types of actions (squaring, multiplying variables, taking variable's reciprocal, etc.) aren't allowed if an equation is **linear**.

Examples of Equations that are or are not Linear

$$2x_1 = 4x_2 - 3x_3 + 5 \quad \text{and} \quad 12 - \sqrt{3}(x + y) = 0$$

These equations are linear.

$$2x_1 - 4x_2 + 3x_3 = 5 \qquad \text{and} \qquad \sqrt{3}x + \sqrt{3}y = 12$$

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Note that both can be written in the format
from the definition. The only operations on
the variables are (1) multiply by constants
and (2) add.
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Examples of Equations that are or are not Linear

$$x_1 + 3x_3 = \frac{1}{x_2} \quad \text{and} \quad xyz = \sqrt{w}$$

These equations are NOT linear.

$\frac{1}{x_2}$ is a nonlinear term

xyz is a nonlinear term

so is $\sqrt{w}$

Definition

A **linear system** (or linear system of equations) is a collection of linear equations in the same variables.

The equations in a linear system are considered together as one *object*.

Example 1:
$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & - & 3x_3 & + & x_4 & = & -3 \\ -x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & = & 8 \end{array}$$

Example 1 is a linear system that has two equations in four variables.

Example 2:
$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4 \\ 3x & & & + & 12z & = & 0 \\ 2x & + & 2y & - & 5z & = & -6 \end{array}$$

Example 2 is a linear system that has three equations in three variables.

In this course, we'll mostly use a single variable name with subscripts, i.e., x_1, x_2, x_3 as opposed to x, y, z.

Some Preliminary Terms

Consider the system of m equations in the variables $x_1, \ldots, x_n$

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & \vdots & \cdots & \vdots & & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array} \tag{1}$$

Definitions: Solution and Solution Set

A **solution** of (1) is an ordered list of numbers $(s_1, s_2, \ldots, s_n)$ that reduce each equation in the system to a true statement upon substitution[a].

The **solutions set** of (1) is the set of all possible solutions.

[a] It is assumed that substitution means setting $x_1 = s_1$, $x_2 = s_2$ and so forth.

Some Preliminary Terms

Consider the system of m equations in the variables $x_1, \ldots, x_n$

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & \vdots & \cdots & \vdots & & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array}$$

Definition: Equivalent Systems

Two linear systems are called **equivalent** (or equivalent systems) if they have the same solution set.

Remark: We'll often use some process to rewrite a system in terms of an equivalent system for which the solution(s) is more obvious.

An Example

$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & -1 \\ -4x_1 & + & 2x_2 & = & 2 \end{array}$$

(a) Show that $(1, 3)$ is a solution.

Set $x_1 = 1$ and $x_2 = 3$

$2(1) - 3 \stackrel{?}{=} -1$

$2 - 3 = -1$ ✓ it's true

$-4(1) + 2(3) \stackrel{?}{=} 2$

$-4 + 6 = 2$ ✓ is also true.

An Example Continued

$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & -1 \\ -4x_1 & + & 2x_2 & = & 2 \end{array}$$

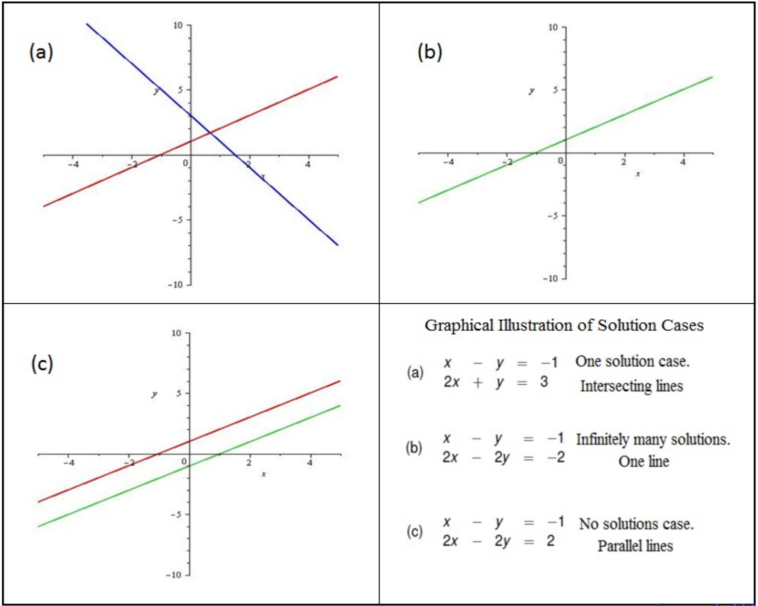
(b) The **solution set** for this system is

$$\left\{ (x_1, x_2) \,|\, x_1 = -\frac{1}{2} + \frac{1}{2}x_2 \right\}.$$

Notice that setting $x_1 = -\frac{1}{2} + \frac{1}{2}x_2$ in each equation we get the pair of true statements

$$\begin{array}{rcrcrl} 2\left(-\frac{1}{2} + \frac{1}{2}x_2\right) & - & x_2 & = & -1, & \quad \text{and} \\ -4\left(-\frac{1}{2} + \frac{1}{2}x_2\right) & + & 2x_2 & = & 2. & \end{array}$$

The Geometry of 2 Equations with 2 Variables



Theorem

Theorem

For a linear system, exactly one of the following holds. The system has

i no solution, or

ii exactly one solution, or

iii infinitely many solutions.

A system is called **inconsistent** if it does not have any solutions (case i), and it's called **consistent** if it has any solution(s) (cases ii & iii).

Note: This theorem speaks to those two big questions:

- Existence: Is there a solution/does a solution exist?
- Uniqueness: Is there a unique solution or multiple solutions?