July 12 Math 2306 sec. 53 Summer 2022

Section 12: LRC Series Circuits

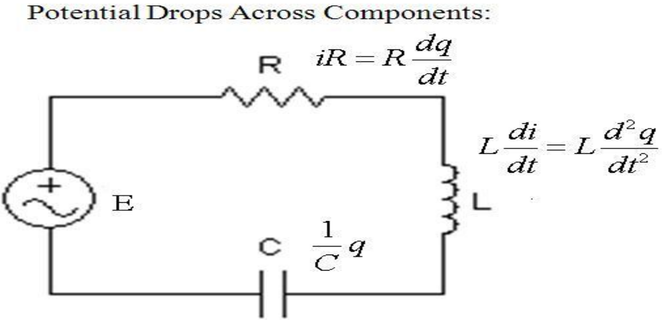

Figure: Kirchhoff's Law: The charge q on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

This is a second order, linear, constant coefficient nonhomogeneous (if $E \neq 0$) equation.

LRC Series Circuit (Free Electrical Vibrations)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free.** These are categorized as

overdamped if	$R^2 - 4L/C > 0$,	(2 real roots)
critically damped if	$R^2 - 4L/C = 0$,	(1 real root)
underdamped if	$R^2 - 4L/C < 0$.	(complex roots)

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function of q_c is influenced by the initial state (q_0 and i_0) and will decay exponentially as $t \to \infty$. Hence q_c is called the **transient state charge** of the system.

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The function q_p is independent of the initial state but depends on the characteristics of the circuit (L, R, and C) and the applied voltage E. q_p is called the **steady state charge** of the system.

Example

An LRC series circuit has inductance 0.5 h, resistance 10 ohms, and capacitance $4 \cdot 10^{-3}$ f. Find the steady state current of the system if the applied force is $E(t) = 5\cos(10t)$.

The ODE for the charge is

$$0.5q'' + 10q' + \frac{1}{4 \cdot 10^{-3}} q = 5\cos(10t) \implies q'' + 20q' + 500q = 10\cos(10t).$$

The characteristic equation $r^2 + 20r + 500 = 0$ has roots $r = -10 \pm 20i$. To determine q_p we can assume

$$q_p = A\cos(10t) + B\sin(10t)$$

which does not duplicate solutions of the homogeneous equation (such duplication would only occur if the roots above were $r = \pm 10i$).

Example Continued...

Working through the details, we find that $A = 1/50$ and $B = 1/100$. The steady state charge is therefore

$$q_p = \frac{1}{50}\cos(10t) + \frac{1}{100}\sin(10t).$$

The steady state current

$$i_p = \frac{dq_p}{dt} = -\frac{1}{5}\sin(10t) + \frac{1}{10}\cos(10t).$$

Section 13: The Laplace Transform

A quick word about functions of 2-variables:

Suppose $G(s,t)$ is a function of two independent variables (s and t) defined over some rectangle in the plane $a \le t \le b$, $c \le s \le d$. If we compute an integral with respect to one of these variables, say t,

$$\int_{\alpha}^{\beta} G(s,t)\,dt$$

- the result is a function of the remaining variable s, and
- the variable s is treated as a constant while integrating with respect to t.

For Example...

Assume that $s \neq 0$ and $b > 0$. Compute the integral

$$\int_0^b e^{-st}\,dt = \frac{1}{-s}e^{-st}\Big|_0^b$$

$$= -\frac{1}{s}e^{-bs} + \frac{1}{s}e^0$$

$$= \frac{1}{s}\left(1 - e^{-bs}\right)$$

This is a function of the variable s.

Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that

$$\int_a^b K(s,t)(\alpha f(t)+\beta g(t))\,dt = \alpha\int_a^b K(s,t)f(t)\,dt + \beta\int_a^b K(s,t)g(t)\,dt.$$

The Laplace Transform

Definition: Let $f(t)$ be defined on $[0, \infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The **kernel** for the Laplace transform is $K(s,t) = e^{-st}$.

Limits at Infinity e^{-st}

If $s > 0$, evaluate

$$\lim_{t\to\infty} e^{-st} = 0$$

If $s > 0$, then $-st \to -\infty$ as $t \to \infty$.

If $s < 0$, evaluate

$$\lim_{t\to\infty} e^{-st} = \infty$$

If $s < 0$, then $-st \to \infty$ as $t \to \infty$.

Find the Laplace transform of $f(t) = 1$

It is readily seen that if $s = 0$, the integral $\int_0^\infty e^{-st}\,dt$ is divergent. Otherwise[1]

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty$$

Convergence in the limit $t \to \infty$ requires $s > 0$. In this case, we have

$$\mathscr{L}\{1\} = -\frac{1}{s}(0-1) = \frac{1}{s}.$$

So we have the transform along with its domain

$$\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0.$$

[1]The integral is improper. We are in reality evaluating an integral of the form $\int_0^b e^{-st} f(t)\,dt$ and then taking the limit $b \to \infty$. We suppress some of the notation here with the understanding that this process is implied.

Find the Laplace transform of $f(t) = t$

Using the definition $\mathscr{L}\{t\} = \int_0^\infty e^{-st}t\,dt$. It's again easy to see that the integral diverges if $s = 0$. Otherwise, we can integrate by parts

$$\int_0^\infty e^{-st}t\,dt = -\frac{1}{s}te^{-st}\Big|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\,dt$$

For $s > 0$, $\lim_{t\to\infty} te^{-st} = 0$. So with the condition that $s > 0$

$$\mathscr{L}\{t\} = 0 + \frac{1}{s}\int_0^\infty e^{-st}\,dt = \frac{1}{s}\mathscr{L}\{1\} = \frac{1}{s^2}$$

So we have the transform along with its domain

$$\mathscr{L}\{t\} = \frac{1}{s^2}, \quad s > 0.$$

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^{10} 2te^{-st}\,dt + \int_{10}^\infty 0 \cdot e^{-st}\,dt$$

For $s \neq 0$, integration by parts gives

$$\mathscr{L}\{f(t)\} = \frac{2}{s^2} - \frac{2e^{-10s}}{s^2} - \frac{20e^{-10s}}{s}.$$

When $s = 0$, the value $\mathscr{L}\{f(t)\}|_{s=10} = 100$.

So we have the transform along with its domain

$$\mathscr{L}\{f(t)\} = \begin{cases} \frac{2}{s^2} - \frac{2e^{-10s}}{s^2} - \frac{20e^{-10s}}{s}, & s \neq 0 \\ 100, & s = 0 \end{cases}$$

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

(a) $f(t) = \cos(\pi t)$

Use $\mathscr{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}, \quad s > 0$

$$\mathscr{L}\{\cos \pi t\} = \frac{s}{s^2+\pi^2}$$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

(b) $f(t) = 2t^4 - e^{-5t} + 3$

$$\begin{aligned}
\mathscr{L}\{2t^4 - e^{-5t} + 3\} &= 2\mathscr{L}\{t^4\} - \mathscr{L}\{e^{-5t}\} + 3\mathscr{L}\{1\} \\
&= 2\frac{4!}{s^{4+1}} - \frac{1}{s-(-5)} + 3\frac{1}{s} \\
&= \frac{48}{s^5} - \frac{1}{s+5} + \frac{3}{s}
\end{aligned}$$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

(c) $f(t) = (2-t)^2$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{4 - 4t + t^2\} = 4\mathscr{L}\{1\} - 4\mathscr{L}\{t\} + \mathscr{L}\{t^2\}$$
$$= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}$$

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Definition: Let $c > 0$. A function f defined on $[0, \infty)$ is said to be of *exponential order c* provided there exists positive constants M and T such that $|f(t)| < Me^{ct}$ for all $t > T$.

Definition: A function f is said to be *piecewise continuous* on an interval $[a, b]$ if f has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Theorem: If f is piecewise continuous on $[0,\infty)$ and of exponential order c for some $c>0$, then f has a Laplace transform for $s>c$.

An example of a function that is NOT of exponential order for any c is $f(t)=e^{t^2}$. Note that

$$f(t)=e^{t^2}=\left(e^t\right)^t \quad \Longrightarrow \quad |f(t)|>e^{ct} \quad \text{whenever} \quad t>c.$$

This is a function that doesn't have a Laplace transform. We won't be dealing with this type of function here.