July 14 Math 2306 sec. 53 Summer 2022

Definition: Let $f(t)$ be defined on $[0,\infty)$. The Laplace transform of f is

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

We'll often use the uppercase-lowercase convention that $F(s) = \mathscr{L}\{f(t)\}$. The domain of $F(s)$ is the set of all s such that the integral is convergent.

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \dots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Section 14: Inverse Laplace Transforms

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ an inverse Laplace transform of $F(s)$.

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \ldots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Using a Table

When using a table of Laplace transforms, the expression must match exactly. For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}$$

so

$$\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} = t^3.$$

Note that $n = 3$, so there must be 3! in the numerator and the power $4 = 3 + 1$ on s.

Find the Inverse Laplace Transform

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\}$

If $n+1=7$, it must be that $n=6$. We need 6! in the numerator. Use the fact that

$$\frac{1}{s^7}=\frac{6!}{s^7}\cdot\frac{1}{6!}.$$

The constant $\frac{1}{6!}$ can be factored out.

$$\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\}=\mathscr{L}^{-1}\left\{\frac{1}{6!}\frac{6!}{s^7}\right\}$$

$$=\frac{1}{6!}\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\}=\frac{t^6}{6!}$$

Example: Evaluate

(b) $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2+9}\right\}$

We just write the argument as the sum of two terms. Since we need $\dfrac{k}{s^2+k^2}$, we can multiply and divide by 3 to get the second term to have the correct format.

$$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= \cos(3t) + \frac{1}{3}\sin(3t)$$

Example: Evaluate

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\}$

First we perform a partial fraction decomposition on the argument. Note that

$$\frac{s-8}{s^2-2s}=\frac{s-8}{s(s-2)}$$

The set up is

$$\frac{s-8}{s(s-2)}=\frac{A}{s}+\frac{B}{s-2}$$

If we clear the fractions, we get

$$s-8=A(s-2)+Bs.$$

Setting $s=0$ we get $-8=-2A$ so $A=4$. Setting $s=2$ we get $-6=2B$ so $B=-3$.

Example: Evaluate

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\}$

Now we can take the inverse transform.

$$\mathscr{L}^{-1}\left\{\frac{s-8}{s^2-2s}\right\} = \mathscr{L}^{-1}\left\{\frac{4}{s} - \frac{3}{s-2}\right\}$$

$$= 4\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 3\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 4 - 3e^{2t}$$

Section 15: Shift Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\}=\frac{2}{s^3}$? Note that by definition

$$\begin{aligned}\mathscr{L}\left\{e^t t^2\right\} &= \int_0^\infty e^{-st}e^t t^2\,dt \\ &= \int_0^\infty e^{-(s-1)t}t^2\,dt\end{aligned}$$

Note: $e^{-st}e^t=e^{-(s-1)t}$

Observe that this is simply the Laplace transform of $f(t)=t^2$ evaluated at $s-1$. Letting $F(s)=\mathscr{L}\left\{t^2\right\}$, we have

$$F(s-1)=\frac{2}{(s-1)^3}.$$

$\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$

If $f(t) = t^2$, then $F(s) = \mathscr{L}\left\{t^2\right\} = \dfrac{2}{s^3}$, and

$$F(s-1) = \frac{2}{(s-1)^3} = \mathscr{L}\left\{e^t t^2\right\}.$$

In general, if $F(s) = \mathscr{L}\{f(t)\}$, then

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt = \int_0^\infty e^{-st}\left[e^{at} f(t)\right] dt$$

$$= \mathscr{L}\left\{e^{at} f(t)\right\}$$

Theorem (translation in s)

Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

In other words, if $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

Example

Evaluate

(a) $\mathscr{L}\{t^6 e^{3t}\} = \dfrac{6!}{(s-3)^7}$

Ignoring the exponential, $F(s) = \mathscr{L}\{t^6\} = \dfrac{6!}{s^7}$ and $a = 3$. The answer is $F(s-3)$.

(b) $\mathscr{L}\{e^{-t}\cos(t)\} \dfrac{s+1}{(s+1)^2+1}$

Ignoring the exponential, $F(s) = \mathscr{L}\{\cos(t)\} = \dfrac{s}{s^2+1}$ and $a = -1$. The answer is $F(s-(-1)) = F(s+1)$.

(c) $\mathscr{L}\{e^{-t}\sin(t)\} \dfrac{1}{(s+1)^2+1}$

Ignoring the exponential, $F(s) = \mathscr{L}\{\sin(t)\} = \dfrac{1}{s^2+1}$ and $a = -1$. The answer is $F(s+1)$.

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\}$

The first idea would be to do a partial fraction decomp. But the quadratic s^2+2s+2 doesn't factor. In this case, we complete the square. Note that

$$s^2+2s+2 = s^2+2s+1-1+2 = (s+1)^2+1.$$

So we have

$$\frac{s}{s^2+2s+2} = \frac{s}{(s+1)^2+1}.$$

This sort of looks like $F(s+1)$ for some function F. However, it would have to have $s+1$ everywhere including the numerator.

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

So that every incidence of s appears as $s+1$, we use the fact that

$$s = s+1-1 \qquad \Longrightarrow \qquad \frac{s}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}.$$

Now we can take the inverse transform

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} = \mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\cos t - e^{-t}\sin t.$$

Observation

We have the statement of the shift in s theorem:

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

Note that from this example

$$\mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} = e^{-1t}\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

and

$$\mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-1t}\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

Doing a partial fraction decomposition, we find that

$$\frac{1+3s-s^2}{s(s-1)^2}=\frac{1}{s}-\frac{2}{s-1}+\frac{3}{(s-1)^2}.$$

So

$$\mathscr{L}^{-1}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\}=\mathscr{L}^{-1}\left\{\frac{1}{s}-\frac{2}{s-1}+\frac{3}{(s-1)^2}\right\}$$

$$=\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}-2\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}+3\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

The first two terms are straightforward. The third term needs some attention.

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\}$

We have the term $\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$. If we replace the $s-1$ with just s, we see that $F(s)=\dfrac{1}{s^2}$ which is the Laplace transform of $f(t)=t$. So using the shifting theorem

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}=e^{1t}\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}=e^t t.$$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

Now we pull it all together to get

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\} &= \\ &= \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} \\ &= 1 - 2e^t + 3te^t\end{aligned}$$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

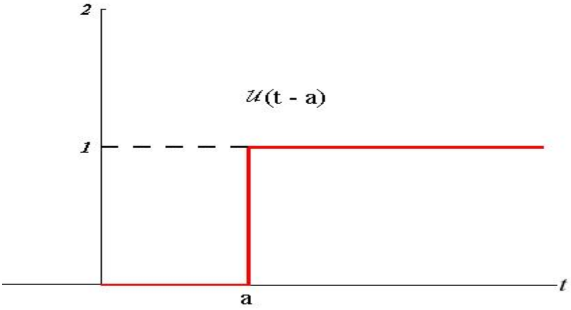

Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

Verify that

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} \; = \; g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

When $0 \le t < a$, the value $\mathscr{U}(t-a) = 0$ (because this is how this function is defined). So the expression on the right side of the equal sign is

$$g(t) - g(t)\cdot 0 + h(t)\cdot 0 = g(t).$$

This is what f is supposed to be when $t < a$. When $t \ge a$, $\mathscr{U}(t-a) = 1$ (again, that's how it's defined). So the expression on the right side of the equal sign is

$$g(t) - g(t)\cdot 1 + h(t)\cdot 1 = h(t).$$

This is also what f is supposed to be.

Piecewise Defined Functions

Verify that

$$f(t) = \begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases} \quad = \; g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

So

$$g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

is another way to write the piecewise defined function

$$f(t) = \begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases}$$

Piecewise Defined Functions in Terms of $\mathscr{U}$

Write f on one line in terms of $\mathscr{U}$ as needed

$$f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ t^2, & 2 \le t < 5 \\ 2t & t \ge 5 \end{cases}$$

We can add in the terms to make them appear and subtract them to make them disappear. Including factors of the unit step function will make sure these occur at the correct time values. We want f to be e^t, but only for $0 \le t < 2$

$$f(t) = e^t - e^t\mathscr{U}(t-2) + \text{something}$$

Piecewise Defined Functions in Terms of $\mathscr{U}$

Write f on one line in terms of $\mathscr{U}$ as needed

$$f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ t^2, & 2 \le t < 5 \\ 2t & t \ge 5 \end{cases}$$

Then we want f to be t^2 for $2 \le t < 5$, so

$$f(t) = e^t - e^t\mathscr{U}(t-2) + t^2\mathscr{U}(t-2) - t^2\mathscr{U}(t-5) + \text{something}$$

and finally, f needs to become $2t$ when $t \ge 5$ (and stay that way for all further t). So

$$f(t) = e^t - e^t\mathscr{U}(t-2) + t^2\mathscr{U}(t-2) - t^2\mathscr{U}(t-5) + 2t\mathscr{U}(t-5)$$

You can look at what this gives you in each interval $0 \le t < 2$, $2 \le t < 5$ and $t \le 5$ to verify that it's correct.

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases} .$$

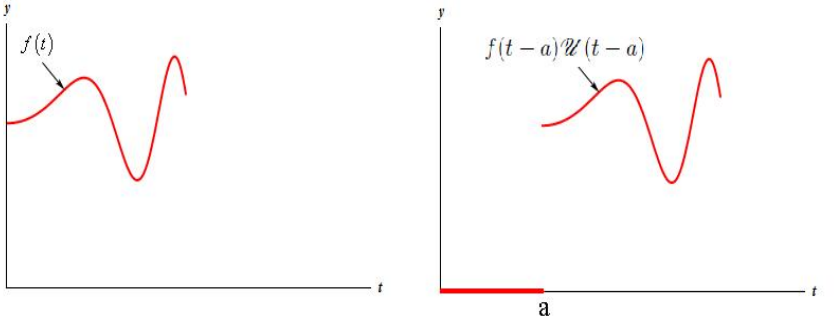

Figure: The function $f(t-a)\mathscr{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a.

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

In particular,

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}.$$

This is just the special case that $f(t) = 1$ so that $F(s) = \frac{1}{s}$.

As another example,

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\{(t-a)^n\mathscr{U}(t-a)\} = \frac{n!e^{-as}}{s^{n+1}}.$$

Find $\mathscr{L}\{\mathscr{U}(t-a)\}$

By definition

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \int_0^\infty e^{-st}\mathscr{U}(t-a)\,dt$$

Break this up into the intervals from 0 to a and from a to infinity.

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}\cdot 1\,dt$$

$$= 0 + \frac{1}{-s}e^{-st}\Big|_a^\infty = -\frac{1}{s}\left(0 - e^{-as}\right)$$

provided $s > 0$ so that the integral converges. This gives the final result

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}.$$

Example

Find the Laplace transform $\mathscr{L}\left\{f(t)\right\}$ where

$$f(t) = \left\{ \begin{array}{ll} 1, & 0 \leq t < 1 \\ t, & t \geq 1 \end{array} \right.$$

(a) First write f in terms of unit step functions.

Using the process from before

$$f(t) = 1 - 1\mathscr{U}(t-1) + t\mathscr{U}(t-1).$$

If we group the last two terms we can write this as

$$f(t) = 1 + (t-1)\mathscr{U}(t-1).$$

Example Continued...

(b) Now use the fact that $f(t) = 1 + (t-1)\mathscr{U}(t-1)$ to find $\mathscr{L}\{f\}$.

Noting that $f(t) = 1 + (t-1)\mathscr{U}(t-1)$, we have

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{1\} + \mathscr{L}\{(t-1)\mathscr{U}(t-1)\}$$

$$= \frac{1}{s} + \frac{e^{-s}}{s^2}.$$

For the second term, note that if $h(t) = t$, then $h(t-1) = t-1$. Hence

$$\mathscr{L}\{h(t-1)\mathscr{U}(t-1)\} = e^{-1s}\mathscr{L}\{h(t)\}.$$

And for $h(t) = t$,

$$\mathscr{L}\{h(t)\} = \mathscr{L}\{t\} = \frac{1!}{s^{1+1}} = \frac{1}{s^2}.$$