July 19 Math 2306 sec. 53 Summer 2022

Section 15: Shift Theorems

We added two shift theorems to our catalog of Laplace transform results.

Theorem (translation in s): Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

In other words, if $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

The Unit Step Function

Then we defined the unit step function $\mathscr{U}(t-a)$ for $a>0$ by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

Theorem (translation in t): If $F(s) = \mathscr{L}\{f(t)\}$ and $a>0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

In other words,

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

Yet another useful statement of this theorem is

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}$$

Example

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}$$

Example: Find $\mathscr{L}\{\cos t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\} = e^{-\frac{\pi}{2}s}\mathscr{L}\{\cos(t+\pi/2)\}$

Note $\cos(t+\pi/2) = \cos t \cos \pi/2 - \sin t \sin \pi/2$

$= \cos t\ (0) - \sin t\ (1)$

$= -\sin t$

$$\Rightarrow \mathcal{L}\{\cos t\, u(t-\pi/2)\} = e^{-\frac{\pi}{2}s}\mathcal{L}\{-\sin t\}$$

$$= -e^{-\frac{\pi}{2}s}\mathcal{L}\{\sin t\}$$

$$= -e^{-\frac{\pi}{2}s}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{-e^{-\pi/2 s}}{s^2+1}$$

Example

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

Example: Find $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+1)}\right\}$

We need to know $f(t) = \mathscr{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\}$

Partial fractions

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \implies 1 = A(s+1) + Bs$$

$s = 0 \implies A = 1$

$s = -1 \implies B = -1$

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = 1 - e^{-t}$$

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = \left(1 - e^{-(t-2)}\right)\mathscr{U}(t-2)$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

Section 16: Laplace Transforms of Derivatives and IVPs

Suppose f has a Laplace transform[1] and that f is differentiable on $[0, \infty)$. Obtain an expression for the Laplace tranform of $f'(t)$ using integration by parts to get

$F(s) = \mathscr{L}\{f(t)\}$

$$\begin{aligned}
\mathscr{L}\left\{f'(t)\right\} &= \int_0^\infty e^{-st} f'(t)\,dt \\
&= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt \\
&= sF(s) - f(0).
\end{aligned}$$

[1] Assume f is of exponential order c for some c.

Transforms of Derivatives

If $\mathscr{L}\{f(t)\} = F(s)$, we have $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$. We can use this relationship recursively to obtain Laplace transforms for higher derivatives of f.

For example

$$\mathscr{L}\{f''(t)\} = s\mathscr{L}\{f'(t)\} - f'(0)$$

$$= s\left(sF(s) - f(0)\right) - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

Transforms of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\{y(t)\} = Y(s),$$

then

$$\begin{aligned}
\mathscr{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0),\\
\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0),\\
&\vdots \\
\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).
\end{aligned}$$

Laplace Transforms and IVPs

For constants a, b, and c, take the Laplace transform of both sides of the equation and isolate $\mathscr{L}\{y(t)\} = Y(s)$.

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Let $G(s) = \mathscr{L}\{g(t)\}$. Take $\mathscr{L}$ of ODE

$$\mathscr{L}\{ay'' + by' + cy\} = \mathscr{L}\{g(t)\}$$

$$a\mathscr{L}\{y''\} + b\mathscr{L}\{y'\} + c\mathscr{L}\{y\} = G(s)$$

$$a(s^2Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = G(s)$$

Isolate $Y(s)$ using algebra

$$as^2Y - asy(0) - ay'(0) + bsY - by(0) + cY = G$$

$$as^2Y - ay_0s - ay_1 + bsY - by_0 + cY = G$$

$$(as^2+bs+c)Y - ay_0s - ay_1 - by_0 = G$$

$$(as^2+bs+c)Y(s) = ay_0s + ay_1 + by_0 + G(s)$$

Characteristic polynomial for the ODE

$$ay'' + by' + cy = g(t),$$

$$Y(s) = \frac{ay_0 s + ay_1 + by_0}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}$$

The solution to the IVP

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}.$$

Solving IVPs

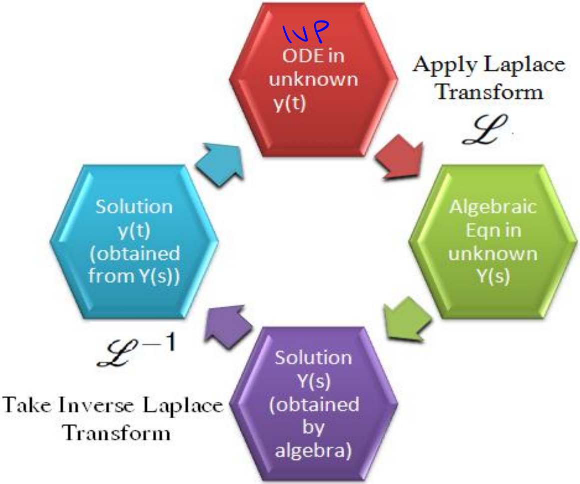


Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

General Form

We get

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where Q is a polynomial with coefficients determined by the initial conditions, G is the Laplace transform of $g(t)$ and P is the **characteristic polynomial** of the original equation.

$$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}$$ is called the **zero input response**,

and

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$$ is called the **zero state response.**

Solve the IVP using the Laplace Transform

$$\frac{dy}{dt}+3y = 2t, \quad y(0) = 2$$

Let $Y(s) = \mathcal{L}\{y(t)\}$.

$$\mathcal{L}\{y' + 3y\} = \mathcal{L}\{2t\}$$

$$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 2\mathcal{L}\{t\}$$

$$sY(s) - y(0) + 3Y(s) = \frac{2}{s^2}$$

$$(s+3)Y(s) - 2 = \frac{2}{s^2}$$

$$(s+3)Y(s) = \frac{2}{s^2} + 2$$

$$Y(s) = \frac{2}{s^2(s+3)} + \frac{2}{s+3}$$

Decompose the 1st term

$$\frac{2}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

$$2 = As(s+3) + B(s+3) + Cs^2$$

$$2 = (A+C)s^2 + (3A+B)s + 3B$$

$$A+C=0, \quad 3A+B=0, \quad 3B=2$$

$$B = \frac{2}{3}, \quad A = -\frac{1}{3}B = \frac{-2}{9} \quad C = -A = \frac{2}{9}$$

$$Y(s) = \frac{-2/9}{s} + \frac{2/3}{s^2} + \frac{2/9}{s+3} + \frac{2}{s+3}$$

$$Y(s) = \frac{-2/9}{s} + \frac{2/3}{s^2} + \frac{20/9}{s+3}$$

The solution to the IVP

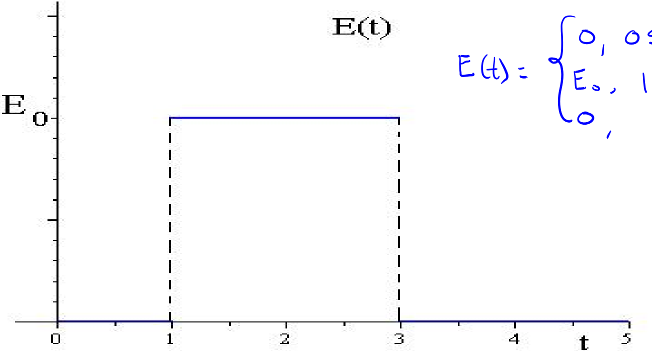
$$y(t) = \frac{-2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{20}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\boxed{y(t) = \frac{-2}{9} + \frac{2}{3}t + \frac{20}{9}e^{-3t}}$$

Solve the IVP

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.

$$L\frac{di}{dt} + Ri = E$$



$$E(t) = \begin{cases} 0, & 0 \le t < 1 \\ E_0, & 1 \le t < 3 \\ 0, & t \geq 3 \end{cases}$$

LR Circuit Example

$$E(t) = 0 - 0\,\mathcal{U}(t-1) + E_0\,\mathcal{U}(t-1) - E_0\,\mathcal{U}(t-3) + 0\,\mathcal{U}(t-3)$$

$$L = 1, \quad R = 10$$

$$\frac{di}{dt} + 10i = E_0\,\mathcal{U}(t-1) - E\,\mathcal{U}(t-3), \quad i(0) = 0$$

Let $I(s) = \mathcal{L}\{i(t)\}$.

$$\mathcal{L}\{i' + 10i\} = \mathcal{L}\{E_0\,\mathcal{U}(t-1) - E_0\,\mathcal{U}(t-3)\}$$

$$\mathcal{L}\{i'\} + 10\,\mathcal{L}\{i\} = E_0\,\mathcal{L}\{\mathcal{U}(t-1)\} - E_0\,\mathcal{L}\{\mathcal{U}(t-3)\}$$

$$sI(s) - \underset{0}{i(0)} + 10\,I(s) = E_0\frac{e^{-s}}{s} - E_0\frac{e^{-3s}}{s}$$

$$(s+10)\,I(s) = E_0\frac{e^{-s}}{s} - E_0\frac{e^{-3s}}{s}$$

$$I(s) = E_0\frac{e^{-s}}{s(s+10)} - E_0\frac{e^{-3s}}{s(s+10)}$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

We need $f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s(s+10)}\right\}$

Using partial fractions

$$\frac{1}{s(s+10)} = \frac{\frac{1}{10}}{s} - \frac{\frac{1}{10}}{s+10}$$

$$f(t) = \mathscr{L}^{-1}\left\{\frac{\frac{1}{10}}{s} - \frac{\frac{1}{10}}{s+10}\right\} = \frac{1}{10} - \frac{1}{10}e^{-10t}$$

$$f(t) = \frac{1}{10} - \frac{1}{10}e^{-10t}$$

$$I(s) = E_0\frac{e^{-s}}{s(s+10)} - E_0\frac{e^{-3s}}{s(s+10)}$$

$$I(s) = \frac{E_0}{10}e^{-s}\left(\frac{1}{s} - \frac{1}{s+10}\right) - \frac{E_0}{10}e^{-3s}\left(\frac{1}{s} - \frac{1}{s+10}\right)$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

$$i(t) = \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right)u(t-1) - \frac{E_0}{10}\left(1 - e^{-10(t-3)}\right)u(t-3)$$

This is the current in the circuit. We can write this as

$$i(t) = \begin{cases} 0, & 0 \le t < 1 \\ \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right), & 1 \le t < 3 \\ \frac{E_0}{10}\left(e^{-10(t-3)} - e^{-10(t-1)}\right), & 3 \le t \end{cases}$$

$0 \le t < 1$	$1 \le t < 3$	$t \ge 3$
$u(t-1) = 0$	$u(t-1) = 1$	$u(t-1) = 1$
$u(t-3) = 0$	$u(t-3) = 0$	$u(t-3) = 1$