July 21 Math 2306 sec. 53 Summer 2022

Section 16: Laplace Transforms of Derivatives and IVPs

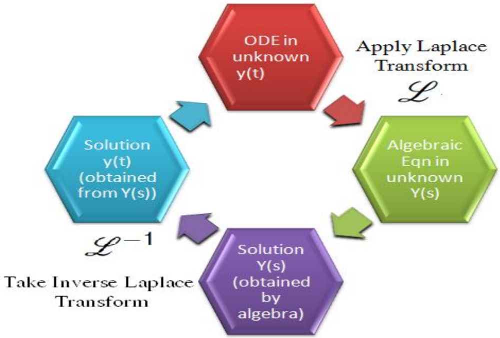

Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

Impulse

A 32 lb object is attached to a spring with spring constant 25 lb/ft. A dashpot induces 8 lb per ft/sec of velocity. The object is at rest at equilibrium. When $t = t_0$, a unit impulse force $f(t) = \delta(t - t_0)$ is applied. Use the fact that $\mathscr{L}\{\delta(t-a)\} = e^{-as}$ for any constant $a \geq 0$ to determine the displacement $x(t)$.

$$mx'' + \beta x' + kx = f(t)$$

weight $W = 32 \text{ lb} = mg = m(32 \text{ ft/sec}^2)$

$\Rightarrow m = 1$ slug

$$x'' + 8x' + 25x = \delta(t - t_0) \quad x(0) = x'(0) = 0$$

$r^2 + 8r + 25$

$$\mathcal{L}\{x'' + 8x' + 25x\} = \mathcal{L}(\delta(t-t_0))$$

$$\mathcal{L}\{x''\} + 8\mathcal{L}\{x'\} + 25\mathcal{L}\{x\} = e^{-t_0 s}$$

Let $X(s) = \mathcal{L}\{x(t)\}$

$$s^2 X(s) - s\underbrace{x(0)}_{=0} - \underbrace{x'(0)}_{=0} + 8\left(sX(s) - \underbrace{x(0)}_{=0}\right) + 25X(s) = e^{-t_0 s}$$

$$(s^2 + 8s + 25)X(s) = e^{-t_0 s}$$

$$\Rightarrow \quad X(s) = \frac{e^{-t_0 s}}{s^2 + 8s + 25}$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,\mathcal{U}(t-a)$$

we need $f(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+8s+25}\right\}$

$s^2+8s+25$ is irreducible $\Rightarrow$ Complete the square

$$s^2+8s+16\quad+25-16 = (s+4)^2+9$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2+9}\right\} = e^{-4t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}$$

$\dfrac{1}{s^2+3^2}$ w/ $s \to s+4$

$$= \frac{1}{3}e^{-4t}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$f(t) = \frac{1}{3} e^{-4t} \sin(3t)$$

$$X(s) = \frac{e^{-t_0 s}}{s^2+8s+25}$$

$$\mathcal{L}^{-1}\{ e^{-as} F(s)\} = f(t-a)\,\mathcal{U}(t-a)$$

$$x(t) = \mathcal{L}^{-1}\{ X(s)\}$$ The displacement

$$\boxed{x(t) = \frac{1}{3} e^{-4(t-t_0)} \sin(3(t-t_0))\,\mathcal{U}(t-t_0)}$$

Solving a System

We can solve a system of ODEs using Laplace transforms. Here, we'll consider systems that are

- linear,
- having initial conditions at $t = 0$, and
- constant coefficient.

Let's see it in action (i.e. with a couple of examples).

Example

Solve the system of equations

$$\begin{aligned}\frac{dx}{dt} &= -2x-2y+60, \quad x(0)=0\\ \frac{dy}{dt} &= -2x-5y+60, \quad y(0)=0\end{aligned}$$

Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$

$$\mathcal{L}\{x'\} = \mathcal{L}\{-2x-2y+60\}$$

$$\mathcal{L}\{y'\} = \mathcal{L}\{-2x-5y+60\}$$

$$sX(s) - x(0) = -2X(s) - 2Y(s) + \mathcal{L}\{60\} \quad \leftarrow 60/s$$

$$x(0) = 0$$

$$sY(s) - y(0) = -2X(s) - 5Y(s) + \frac{60}{s}$$

$$y(0) = 0$$

$$sX + 2X + 2Y = \frac{60}{s} \qquad (s+2)X + 2Y = \frac{60}{s}$$

$$sY + 2X + 5Y = \frac{60}{s} \qquad 2X + (s+5)Y = \frac{60}{s}$$

$$\begin{bmatrix} s+2 & 2 \\ 2 & s+5 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 60/s \\ 60/s \end{bmatrix}$$

$$A = \begin{bmatrix} s+2 & 2 \\ 2 & s+5 \end{bmatrix}, \quad A_x = \begin{bmatrix} 60/s & 2 \\ 60/s & s+5 \end{bmatrix}, \quad A_y = \begin{bmatrix} s+2 & 60/s \\ 2 & 60/s \end{bmatrix}$$

$$\det(A) = (s+2)(s+5) - 2\cdot 2$$

$$= s^2 + 7s + 10 - 4$$

$$= s^2 + 7\cdot s + 6 \quad = (s+1)(s+6)$$

$$\det(A_x) = \frac{60}{s}(s+5) - \frac{60}{s}(2) = 60 + \frac{60}{s}(5-2)$$

$$= 60 + \frac{60}{s}\cdot 3$$

$$\det(A_y) = (s+2)\frac{60}{s} - 2\cdot\frac{60}{s} = 60 + \frac{60}{s}(2-2)$$

$$X(s) = \frac{\det(A_x)}{\det(A)} = \frac{60\left(1+\frac{3}{s}\right)}{(s+1)(s+6)} = \frac{60(s+3)}{s(s+1)(s+6)}$$

$$Y(s) = \frac{\det(A_y)}{\det(A)} = \frac{60}{(s+1)(s+6)}$$

We need to do partial fraction decomps.

$$X(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+6} = \frac{30}{s} - \frac{24}{s+1} - \frac{6}{s+6}$$

$$Y(s) = \frac{D}{s+1} + \frac{E}{s+6} = \frac{12}{s+1} - \frac{12}{s+6}$$

Finally

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{30}{s} - \frac{24}{s+1} - \frac{6}{s+6}\right\}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{12}{s+1} - \frac{12}{s+6}\right\}$$

$$\boxed{\begin{aligned} x(t) &= 30 - 24e^{-t} - 6e^{-6t} \\ y(t) &= 12e^{-t} - 12e^{-6t} \end{aligned}}$$