July 26 Math 2306 sec. 53 Summer 2022

Some Random Review Problems

Solve the IVP.

$$x\frac{dy}{dx}+3y=\frac{1}{1+x^3},\quad y(1)=0$$

The ODE is first order linear. The general solution

$$y=\frac{1}{3x^3}\ln(1+x^2)+\frac{C}{x^3},$$

and the solution to the IVP

$$y=\frac{\ln(1+x^2)-\ln(2)}{3x^3}.$$

Use Cramer's rule to solve the system for X and Y.

$$\begin{array}{ccccc} sX & - & 4Y & = & 1 \\ X & + & sY & = & \frac{2}{s} \end{array}$$

Let $A = \begin{bmatrix} s & -4 \\ 1 & s \end{bmatrix}$,

$$A_X = \begin{bmatrix} 1 & -4 \\ \frac{2}{s} & s \end{bmatrix}, \quad \text{and} \quad A_Y = \begin{bmatrix} s & 1 \\ 1 & \frac{2}{s} \end{bmatrix}.$$

Then

$$X(s) = \frac{\det(A_X)}{\det(A)} = \frac{s^2+8}{s(s^2+4)} \quad \text{and} \quad Y(s) = \frac{\det(A_Y)}{\det(A)} = \frac{1}{s^2+4}$$

Solve the system of IVPs

$$\begin{array}{rcrcl} x'(t) & - & 4y(t) & = & 0, \qquad x(0) = 1 \\ x(t) & + & y'(t) & = & 2, \qquad y(0) = 0 \end{array}$$

Take the Laplace transform of both equations. Let $X(s) = \mathscr{L}\{x(t)\}$ and $Y(s) = \mathscr{L}\{y(t)\}$

$$\begin{array}{ccccc} sX(s) - x(0) & - & 4Y(s) & = & 0 \\ X(s) & + & sY(s) - y(0) & = & \frac{2}{s} \end{array}$$

Plugging in the initial conditions and isolating (X, Y) on the left gives the system

$$\begin{array}{ccccc} sX & - & 4Y & = & 1 \\ X & + & sY & = & \frac{2}{s} \end{array}$$

Continuation...

This is the system from the last example. We need a partial fraction decomp on X in order to take the inverse transform (Y is ready as is).

$$\begin{aligned} X(s) &= \frac{s^2+8}{s(s^2+4)} = \frac{2}{s} - \frac{s}{s^2+4} \\ Y(s) &= \frac{1}{s^2+4} \end{aligned}$$

Taking the inverse transform

$$\begin{aligned} x(t) &= 2 - \cos(2t) \\ y(t) &= \frac{1}{2}\sin(2t) \end{aligned}$$

Solve the IVP

$$\frac{dy}{dt} = \frac{y}{\sqrt{1-t^2}}, \quad y(0) = 2$$

The ODE is both separable and linear. The solution to the IVP

$$y = 2e^{\sin^{-1}(t)}.$$

Solve the IVP using the Laplace Transform

$$y'' + y' - 6y = 30e^{-t}, \quad y(0) = 2, \quad y'(0) = -1$$

Letting $Y(s) = \mathscr{L}\{y(t)\}$, taking the transform of the ODE, inserting the IC, and solving for Y gives

$$Y(s) = \frac{30}{(s+1)(s+3)(s-2)} + \frac{2s+1}{(s+3)(s-2)}.$$

This is the same as $\dfrac{2s^2+3s+31}{(s+1)(s+3)(s-2)}$. Some partial fraction decomp(s) will be needed to take the inverse transform.

$$Y(s) = \frac{3}{s-2} + \frac{4}{s+3} - \frac{5}{s+1}$$

and the solution

$$y(t) = 3e^{2t} + 4e^{-3t} - 5e^{-t}.$$

Find the general solution.

$$y'' + y' - 6y = 30e^{-t}$$

There are options on how to approach this. Without initial conditions, Laplace transforms aren't really the way to go. (They could be used, but it would be necessary to introduce IC in the form $y(0) = y_0$ and $y'(0) = y_1$.)

$$y = c_1 e^{2t} + c_2 e^{-3t} - 5e^{-t}.$$

Solve the IVP

$$x\frac{dy}{dx} + y = \frac{4x^2}{y}, \quad y(1) = 1$$

The ODE is a Bernoulli equation with $n = -1$ (using the notation from earlier). The general solution

$$y = \sqrt{2x^2 + \frac{C}{x^2}}.$$

The solution to the IVP

$$y = \sqrt{2x^2 - \frac{1}{x^2}}.$$

Solve the IVP[1]

$$y'' + 2y' + 26y = \delta(t-3), \quad y(0) = 2, \quad y'(0) = 3$$

Letting $Y(s) = \mathscr{L}\{y(t)\}$,

$$Y(s) = \frac{e^{-3s}}{s^2 + 2s + 26} + \frac{2s+7}{s^2 + 2s + 26}.$$

This requires completing the square to make taking the inverse transform possible.

$$Y(s) = \frac{1}{5}\frac{5e^{-3s}}{(s+1)^2 + 5^2} + \frac{2(s+1)}{(s+1)^2 + 5^2} + \frac{5}{(s+1)^2 + 5^2}.$$

The solution to the IVP

$$y(t) = \frac{1}{5}e^{-(t-3)}\sin(5(t-3))\mathscr{U}(t-3) + 2e^{-t}\cos(5t) + e^{-t}\sin(5t).$$

[1] $\mathscr{L}\{\delta(t-a)\} = e^{-as}$

Determine the form of the particular solution.

$$y'' + 2y' + 26y = 3xe^{-x}\cos(5x) - 5x^3$$

The complementary solution is $y_c = c_1 e^{-t}\cos(5t) + c_2 e^{-t}\sin(5t)$. Using the principle of superposition, let y_{p_1} solve $y'' + 2y' + 26y = 3xe^{-x}\cos(5x)$ and y_{p_2} solve $y'' + 2y' + 26y = -5x^3$. Working through the details

$$y_{p_1} = (Ax^2 + Bx)e^{-x}\cos(5x) + (Cx^2 + Dx)e^{-x}\sin(5x), \quad \text{and}$$

$$y_{p_2} = Ex^3 + Fx^2 + Gx + H.$$

The particular solution to the whole problem would have the form

$$y_p = (Ax^2+Bx)e^{-x}\cos(5x)+(Cx^2+Dx)e^{-x}\sin(5x)+Ex^3+Fx^2+Gx+H.$$

Find the general solution.

One solution to the homogeneous equation is $y_1 = x$.

$$x^3y'' + xy' - y = 3e^{1/x}$$

Using reduction of order, the second solution to the homogeneous equation would be $y_2 = xe^{1/x}$. Then using variation of parameters, $y_p = -3xe^{1/x} + 3e^{1/x}$. The general solution

$$y = c_1x + c_2xe^{1/x} + 3e^{1/x}.$$

Solve the IVP

$$y'' - 4y' + 13y = 0, \quad y(0) = 1, \quad y'(0) = 3$$

The general solution

$$y = c_1 e^{2x} \cos(3x) + c_2 e^{2x} \sin(3x),$$

and the solution to the IVP

$$y = e^{2x} \cos(3x) + \frac{1}{3} e^{2x} \sin(3x).$$