July 5 Math 2306 sec. 53 Summer 2022

Section 10: Variation of Parameters

We are still considering nonhomogeneous, linear ODEs. Consider equations of the form

$$y'' + y = \tan x, \quad \text{or} \quad x^2y'' + xy' - 4y = e^x.$$

The method of undetermined coefficients is not applicable to either of these.

- The first equation has constant coefficient left side, but the tangent is not the right kind of right hand side.
- The second equation has an exponential right side, but the left side isn't constant coefficient.

We need another approach.

Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

$y_c = c_1 y_1 + c_2 y_2$

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

$y_p = u_1 y_1 + u_2 y_2$

$y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$

Let's assume $\quad u_1' y_1 + u_2' y_2 = 0$

$y_p' = u_1 y_1' + u_2 y_2'$

$y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

$$y_p'' + P(x)y_p' + Q(x)y_p = g(x)$$

$$u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2' + P(x)(u_1y_1' + u_2y_2') + Q(x)(u_1y_1 + u_2y_2) = g(x)$$

Collect u_1, u_2, u_1', u_2'

$$\underbrace{(y_1'' + P(x)y_1' + Q(x)y_1)}_{0}u_1 + \overbrace{(y_2'' + P(x)y_2' + Q(x)y_2)}^{=0}u_2 + u_1'y_1' + u_2'y_2' = g(x)$$

2nd equation

$$u_1'y_1' + u_2'y_2' = g(x)$$

We have two equations for u_1 and u_2

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = g(x)$$

We'll solve using Cramer's rule.

In matrix format, the system is

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

Wronskian matrix

Let $W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}$, $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$u_1' = \frac{-g\,y_2}{W}, \quad u_2' = \frac{y_1\,g}{W}$$

$$u_1 = \int \frac{-g\,y_2}{W}\,dx \quad \text{and} \quad u_2 = \int \frac{g\,y_1}{W}\,dx$$

Then $y_p = u_1 y_1 + u_2 y_2$

Example:

Solve the ODE $y'' + y = \tan x$.

Find y_c : $y'' + y = 0$

Characteristic eqn $m^2 + 1 = 0 \implies m = \pm i$

$\alpha = 0 , \beta = 1$

$y_1 = \cos x$, $y_2 = \sin x$

Compute the Wronskian:

$$W(y_1, y_2)(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

The equation is in standard form, $g(x) = \tan x$

we have, $y_1 = \cos x$, $y_2 = \sin x$, $g(x) = \tan x$

$W = 1$.

Set $y_p = u_1 y_1 + u_2 y_2$ where

$$u_1 = \int \frac{-g y_2}{W}\, dx = \int -\frac{\tan x \sin x}{1}\, dx$$

$$= \int (\cos x - \sec x)\, dx = \sin x - \ln|\sec x + \tan x|$$

$$u_2 = \int \frac{g y_1}{W}\, dx = \int \frac{\tan x \cos x}{1}\, dx$$

$$= \int \sin x\, dx = -\cos x$$

$y_1 = \cos x, \quad y_2 = \sin x, \quad u_1 = \sin x - \ln|\sec x + \tan x|$

$u_2 = -\cos x$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \cos x\left(\sin x - \ln|\sec x + \tan x|\right) + \sin x\,(-\cos x)$$

$$= \cos x \sin x - \cos x \ln|\sec x + \tan x| - \sin x \cos x$$

$$\boxed{y_p = -\cos x \ln|\sec x + \tan x|}$$

The general solution

$$\boxed{y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|}$$

Example:

Solve the ODE

$$x^2y'' + xy' - 4y = \ln x,$$

given that $y_c = c_1x^2 + c_2x^{-2}$ is the complementary solution.

From y_c given

$y_1 = x^2 \qquad y_2 = x^{-2}$

The Wronskian $W = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = -2x^2x^{-3} - 2x\,x^{-2}$

$= -4x^{-1}$

we need $g(x)$: In standard form,

$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = \frac{\ln x}{x^2} \qquad g(x) = \frac{\ln x}{x^2}$

We have $y_1 = x^2$, $y_2 = x^{-2}$, $W = -4x^{-1}$, $g(x) = \frac{\ln x}{x^2}$.

Setting $y_p = u_1 y_1 + u_2 y_2$

$$u_1 = \int \frac{-g y_2}{W}\,dx = \int \frac{-\frac{\ln x}{x^2}\cdot x^{-2}}{-4x^{-1}}\,dx$$

$$= \frac{1}{4}\int \ln x\,(x^{-3})\,dx = \frac{1}{4}\int x^{-3}\ln x\,dx$$

$$u_1 = \frac{-1}{8}x^{-2}\ln x - \frac{1}{16}x^{-2}$$

$$u_2 = \int \frac{g y_1}{W}\,dx = \int \frac{\frac{\ln x}{x^2}\,x^2}{-4x^{-1}}\,dx = \frac{-1}{4}\int x\ln x\,dx.$$

$u_2 = \frac{-1}{8}x^2\ln x + \frac{1}{16}x^2$

$y_1 = x^2,\ y_2 = x^{-2}$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\frac{-1}{8}x^{-2}\ln x - \frac{1}{16}x^{-2}\right)x^2 + \left(\frac{-1}{8}x^2\ln x + \frac{1}{16}x^2\right)x^{-2}$$

$$= \frac{-1}{8}\ln x - \frac{1}{16} - \frac{1}{8}\ln x + \frac{1}{16}$$

$$= \frac{-1}{4}\ln x$$

The general solution

$$\boxed{y = C_1 x^2 + C_2 x^{-2} - \frac{1}{4}\ln x}$$

Solve the IVP

$$x^2y'' + xy' - 4y = \ln x, \quad y(1) = -1, \quad y'(1) = 0$$

We found the general solution

$$y = c_1 x^2 + c_2 x^{-2} - \frac{1}{4}\ln x$$

Apply the initial conditions

$$y' = 2c_1 x - 2c_2 x^{-3} - \frac{1}{4}\frac{1}{x}$$

$$y(1) = c_1(1^2) + c_2(1^{-2}) - \frac{1}{4}\ln 1 = -1$$

$$c_1 + c_2 = -1$$

$$y'(1) = 2c_1(1) - 2c_2(1^{-3}) - \frac{1}{4}\frac{1}{1} = 0$$

$$2c_1 - 2c_2 = \frac{1}{4}$$

$$c_1 + c_2 = -1 \qquad 2c_1 + 2c_2 = -2$$

$$2c_1 - 2c_2 = \frac{1}{4} \qquad 2c_1 - 2c_2 = \frac{1}{4}$$

add $\quad 4c_1 = \frac{-7}{4} \implies c_1 = \frac{-7}{16}$

subtract $\quad 4c_2 = \frac{-9}{4} \implies c_2 = \frac{-9}{16}$

$$\boxed{\text{The solution to the IVP is} \quad y = \frac{-7}{16}x^2 - \frac{9}{16}x^{-2} - \frac{1}{4}\ln x}$$