July 7 Math 2306 sec. 53 Summer 2022

Section 11: Linear Mechanical Equations

Simple Harmonic Motion

We consider a flexible spring from which a mass is suspended. In the absence of any damping forces (e.g. friction, a dash pot, etc.), and free of any external driving forces, any initial displacement or velocity imparted will result in **free, undamped motion**–a.k.a. **simple harmonic motion**.

Harmonic Motion gif

Building an Equation: Hooke's Law

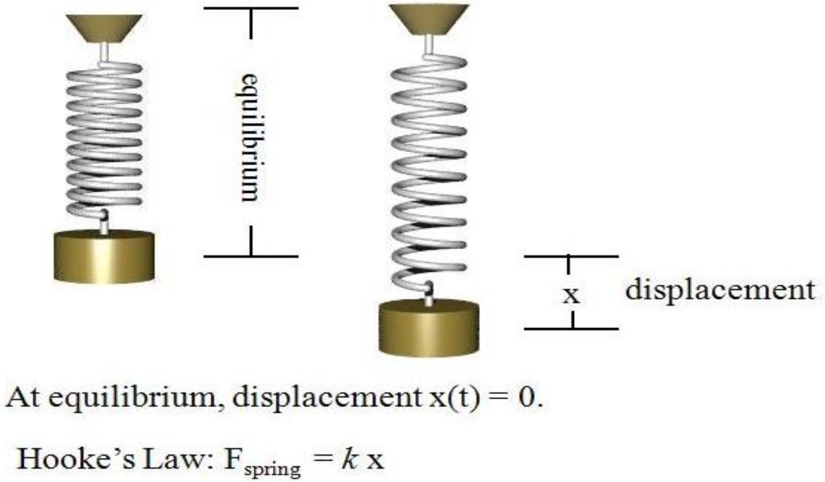

Figure: In the absence of any displacement, the system is at equilibrium. Displacement $x(t)$ is measured from equilibrium $x = 0$.

Building an Equation: Hooke's Law

Newton's Second Law: $F = ma$ (mass times acceleration)

$$a = \frac{d^2x}{dt^2} \implies F = m\frac{d^2x}{dt^2}$$

Hooke's Law: $F = kx$ (proportional to displacement)

Equate the forces

$$m\frac{d^2x}{dt^2} = -kx \Rightarrow mx'' + kx = 0$$

2nd order, linear, homogeneous, constant coef. ODE. In standard form

$$x'' + \omega^2 x = 0 \qquad \omega = \sqrt{\frac{k}{m}}$$

Displacment in Equilibrium

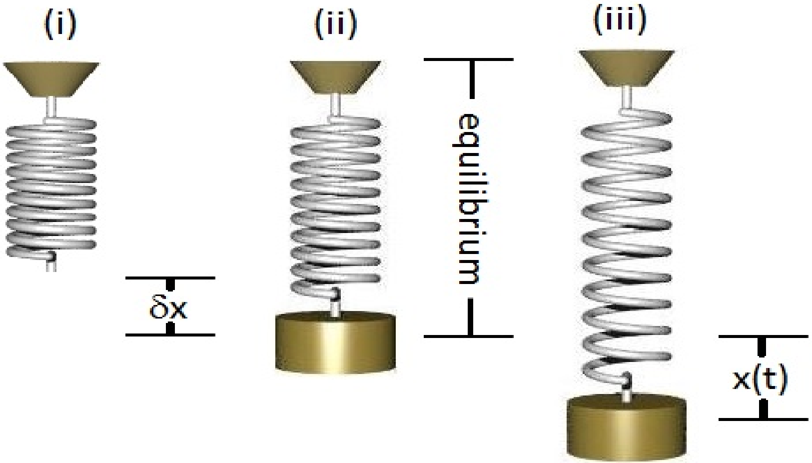

Figure: Spring only, versus spring-mass equilibrium, and spring-mass (nonzero) displacement

Obtaining the Spring Constant (US Customary Units)

If an object with weight W pounds stretches a spring δx feet in equilibrium, then by Hooke's law we compute the spring constant via the equation

$$W = k\delta x.$$

The units for k in this system of measure are lb/ft.

$$k = \frac{W}{\delta x} \qquad \frac{\text{lb}}{\text{ft}}$$

Obtaining the Spring Constant (US Customary Units)

Note also that Weight = mass $\times$ acceleration due to gravity. Hence if we know the weight of an object, we can obtain the mass via

$$W = mg.$$

We typically take the approximation $g = 32$ ft/sec^2. The units for mass are lb sec^2/ft which are called slugs.

$$m = \frac{W}{g}. \quad \text{slugs}$$

Obtaining the Spring Constant (SI Units)

In SI units,

- Weight (force) would be in Newtons (N),
- Length would be in meters (m),
- Spring constant would be in N/m
- Mass would be in kilograms (kg)

It is customary to describe an object by its mass in kilograms. When we encounter such a description, we deduce the weight in Newtons

$$W = mg \quad \text{taking the approximation} \quad g = 9.8\,\mathrm{m}/\mathrm{sec}^2.$$

The *Circular Frequency* ω

Applying Hooke's law with the weight as force, we have

$$mg = k\delta x. \qquad \Rightarrow \frac{mg}{m\delta x} = \frac{k\delta x}{m\delta x}$$

We observe that the value ω can be deduced from δx by

$$\omega^2 = \frac{k}{m} = \frac{g}{\delta x}.$$

Provided that values for δx and g are used in appropriate units, ω is in units of per second.

Simple Harmonic Motion

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{1}$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) \tag{2}$$

called the **equation of motion.**

Convention
up positive
down = negative

Caution: The phrase **equation of motion** is used differently by different authors.

Some use this phrase to refer the IVP (1). Others use it to refer to the **solution** to the IVP such as (2).

Simple Harmonic Motion

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$[1]
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[1] Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by

$$\sin \phi = \frac{x_0}{A}, \quad \text{with} \quad \cos \phi = \frac{x_1}{\omega A}.$$

Amplitude and Phase Shift (alternative definition)

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \cos(\omega t - \hat{\phi})$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and this **phase shift** $\hat{\phi}$ must be defined by

$$\cos \hat{\phi} = \frac{x_0}{A}, \quad \text{with} \quad \sin \hat{\phi} = \frac{x_1}{\omega A}.$$

Example

An object stretches a spring 6 inches in equilibrium. Assuming no driving force and no damping, set up the differential equation describing this system.

The ODE is of the form

$$x'' + \omega^2 x = 0$$

we're given displacement in equilibrium

$$\delta x = 6 \text{ in.}$$

we can use the formula $\omega^2 = \dfrac{g}{\delta x}$

$\delta x = \frac{1}{2}$ ft $\qquad g = 32$ ft/sec^2

$$\omega^2 = \frac{32 \ ft/sec^2}{\frac{1}{2} \ ft} = 64 \ \frac{1}{sec^2}$$

$$x'' + 64x = 0$$ is the ODE

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec^2.)

Let's compute k and m.

The weight $W = 4\,\text{lb}$. Here $\delta x = \frac{1}{2}\,\text{ft}$

$$W = k\,\delta x \implies k = \frac{4\,\text{lb}}{\frac{1}{2}\,\text{ft}} = 8\,\frac{\text{lb}}{\text{ft}}$$

The mass:

$$W = mg \implies m = \frac{4\,\text{lb}}{32\,\frac{\text{ft}}{\text{sec}^2}} = \frac{1}{8}\ \text{slugs}$$

$$\omega^2 = \frac{k}{m} = \frac{8\ \frac{\text{lb}}{\text{ft}}}{\frac{1}{8}\ \text{slugs}} = 64\ \frac{1}{\text{sec}^2}$$

The IVP is

$$x'' + 64x = 0, \quad x(0) = 4 \quad x'(0) = -24$$

The characteristic eqn is

$$r^2 + 64 = 0 \Rightarrow r^2 = -64 \Rightarrow r = \pm 8i$$

$$x = c_1 \cos(8t) + c_2 \sin(8t)$$

$$x' = -8c_1 \sin(8t) + 8c_2 \cos(8t)$$

$$x(0) = c_1 \cos(0) + c_2 \sin(0) = 4 \Rightarrow c_1 = 4$$

$$x'(0) = -8c_1 \sin(0) + 8c_2 \cos(0) = -24 \Rightarrow c_2 = \frac{-24}{8} = -3$$

The equation of motion

$$\boxed{x = 4\cos(8t) - 3\sin(8t)}$$

The period $T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$

The frequency $f = \frac{1}{T} = \frac{4}{\pi}$

$$\omega = 8$$

The amplitude $A = \sqrt{4^2 + (-3)^2} = 5$

$$x = 5\sin(8t + \phi) \quad \text{where}$$

$$\sin\phi = \frac{x_0}{A} = \frac{4}{5} \quad \text{and} \quad \cos\phi = \frac{x_1}{\omega A} = \frac{-3}{5}$$

$$\phi = \cos^{-1}\left(\frac{-3}{5}\right) = 2.21 \qquad \phi \approx 127^\circ$$

The phase shift is

about 2.21 rad.

Free Damped Motion

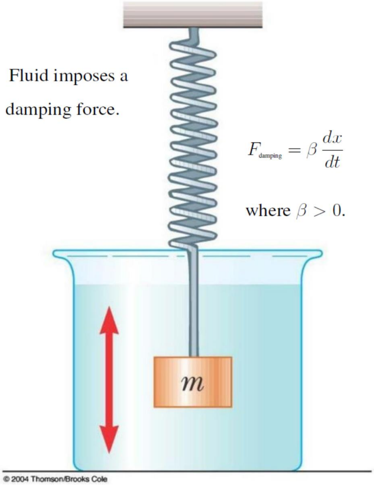

Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

$$\text{Total Force} = \text{Force of spring} + \text{Force of damping}$$

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

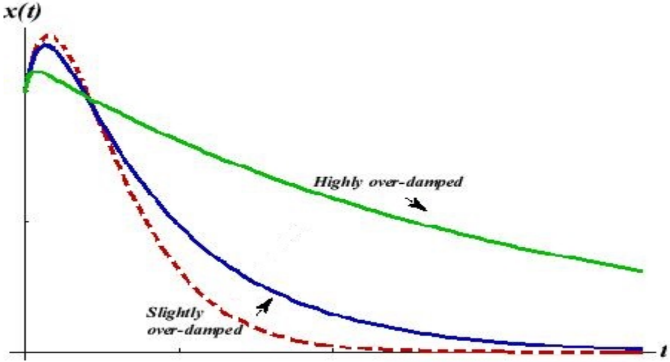

Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

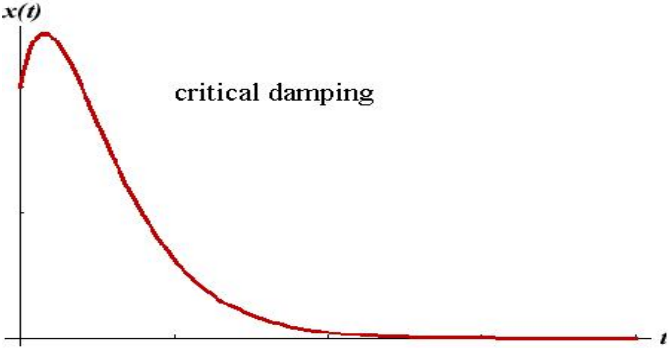

Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

$$x(t) = e^{-\lambda t}\left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

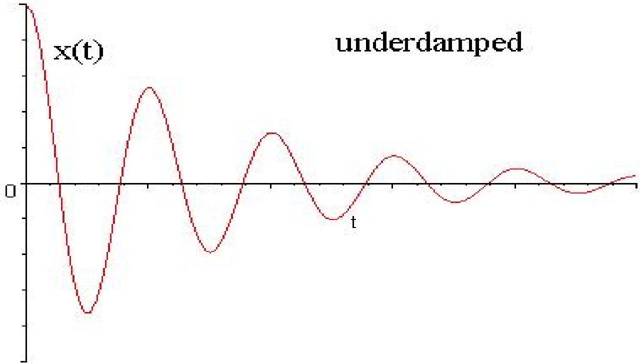

Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping

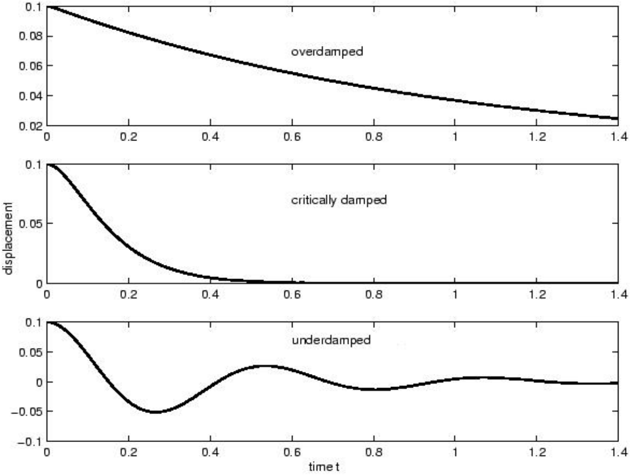

Figure: Comparison of motion for the three damping types.

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

The ODE is

$$mx'' + \beta x' + kx = 0$$

$$m = 2, \quad k = 12, \quad \beta = 10$$

$$2x'' + 10x' + 12x = 0$$

In standard form: $x'' + 5x' + 6x = 0$

Characteristic eqn. $r^2 + 5r + 6 = 0$

This factors $(r+2)(r+3) = 0$

$\Rightarrow r = -2$ or $r = -3$

Since there are two distinct real roots, the system is overdamped.

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

The ODE is $mx'' + \beta x' + kx = 0$

$m = 3, \quad \beta = 12, \quad k = 12$

So $3x'' + 12x' + 12x = 0$

In standard form $x'' + 4x' + 4x = 0$

The characteristic eqn

$$r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0$$

$r = -2$ repeated

The system is critically damped

The solution is

$$x = c_1 e^{-2t} + c_2 t e^{-2t}$$

From the statement, $x(0) = 0$ and $x'(0) = 1$

$$x'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

$$x(0) = c_1 e^{0} + c_2 \cdot 0 e^{0} = 0 \Rightarrow c_1 = 0$$

$$x'(0) = c_2 e^{0} - 2c_2 \cdot 0 e^{0} = 1 \Rightarrow c_2 = 1$$

The displacement

$$\boxed{x(t) = t e^{-2t}}$$

Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

2nd order linear constant coef, nonhomogeneous

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and (2)} \quad \gamma = \omega.$$

Taking the sine case, the DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

$x'' + \omega^2 x = F_0 \sin(\gamma t)$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

Suppose $\gamma \neq \omega$

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$

This is the correct form since $\gamma \neq \omega$.

The gen. solution would be

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t) + A\cos(\gamma t) + B\sin(\gamma t)$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

Suppose $\gamma = \omega$

$$x_p = A\cos(\gamma t) + B\sin(\overset{\omega}{\gamma} t)$$

This matches x_c. The correct form is

$$x_p = At\cos(\omega t) + Bt\sin(\omega t)$$

The general soln.

$$x = c_1\cos(\omega t) + c_2\sin(\omega t) + At\cos(\omega t) + Bt\sin(\omega t)$$

Forced Undamped Motion and Resonance

For $F(t) = F_0 \sin(\gamma t)$ starting from rest at equilibrium:

from rest

$$\text{Case (1):} \quad x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2}\left(\sin(\gamma t) - \frac{\gamma}{\omega}\sin(\omega t)\right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

$$\text{Case (2):} \quad x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0}{2\omega} t \cos(\omega t)$$

Note that the amplitude, α, of the second term is a function of t:

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

Forced Motion and Resonance Applet

Choose ”Elongation diagram” to see a plot of displacement. Try exciter frequencies close to ω.