July 11 Math 3260 sec. 51 Summer 2023

Section 4.5: Dimension of a Vector Space

Theorem:

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set of vectors in V containing *more than n vectors* is linearly dependent.

Corollary:

If vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then every basis of V consist of exactly n vectors.

Dimension of a Vector Space V

Definition:

If V is spanned by a finite set, then V is called **finite dimensional**. In this case, the dimension of V

$$\dim V \;=\; \text{the number of vectors in any basis of } V.$$

The dimension of the vector space $\{\mathbf{0}\}$ containing only the zero vector is defined to be zero—i.e.

$$\dim\{\mathbf{0}\} = 0.$$

If V is not spanned by a finite set[a], then V is said to be **infinite dimensional**.

[a] $C^0(\mathbb{R})$ is an example of an infinite dimensional vector space.

Subspaces and Dimension

Theorem:

Let H be a subspace of a finite dimensional vector space V. Then H is finite dimensional and

$$\dim H \leq \dim V.$$

Moreover, any linearly independent subset of H can be expanded if needed to form a basis for H.

Theorem:

Let V be a vector space with $\dim V = p$ where $p \geq 1$. Any linearly independent set in V containing exactly p vectors is a basis for V. Similarly, any spanning set consisting of exactly p vectors in V is necessarily a basis for V.

Column and Null Spaces

Theorem:

Let A be an $m \times n$ matrix. Then

$$\dim \operatorname{Nul} A = \text{the number of free variables in the equation } A\mathbf{x} = \mathbf{0},$$

and

$$\dim \operatorname{Col} A = \text{the number of pivot positions in } A.$$

Remarks

- Row operations preserve row space, but change linear dependence relations of rows.
- Row operations change column space, but preserve linear dependence relations of columns.
- Another way to obtain a basis for $\mathrm{Row}\, A$ is to take the transpose A^T and do row operations. We have the following relationships:

$$\mathrm{Row}\, A = \mathrm{Col}\, A^T \quad \text{and} \quad \mathrm{Col}\, A = \mathrm{Row}\, A^T$$

Rank & Nullity

Definition:

The **rank** of a matrix A, denoted $\text{rank}(A)$, is the dimension of the column space of A.

Definition:

The **nullity** of a matrix A is the dimension of the null space.

Remark: Since the dimension of the column space is the number of pivot positions, the dimensions of the column and row spaces are the same. That is,

$$\text{rank}(A) = \dim \text{Col}(A) = \dim \text{Row}(A).$$

The Rank-Nullity Theorem

Theorem:

For $m \times n$ matrix A, $\dim \operatorname{Col}(A) = \dim \operatorname{Row}(A) = \operatorname{rank}(A)$. Moreover

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

Note: This theorem states the rather obvious fact that

$$\left\{ \begin{array}{c} \text{number of} \\ \text{pivot columns} \end{array} \right\} + \left\{ \begin{array}{c} \text{number of} \\ \text{non-pivot columns} \end{array} \right\} = \left\{ \begin{array}{c} \text{total number} \\ \text{of columns} \end{array} \right\}.$$

Examples

(1) A is a 5×4 matrix and $\text{rank}(A) = 4$. What is $\dim \text{Nul}\, A$?

rank + nullity = n

Here $n = 4$, $\text{rank}(A) = 4$

$\dim(\text{Nul}\, A) = 4 - 4 = 0$

$A\vec{x} = \vec{0}$ has no nontrivial solutions.

Examples

(2) Suppose A is 7×5 and $\dim \operatorname{Col} A = 2$. Determine the nullity of A.

rank + nullity = n

$n = 5$, $\operatorname{rank}(A) = \dim(\operatorname{Col} A) = 2$

nullity $= 5 - 2 = 3$

Examples

(3) Suppose A is 7×5 and $\dim \mathrm{Col}\, A = 2$. Determine

1. the rank of A^T

$$\mathrm{rank}(A^T) = \dim(\mathrm{Col}\, A^T)$$
$$= \dim(\mathrm{Row}\, A) = 2$$

2. the nullity of A^T

A^T is $5 \times 7 \Rightarrow n = 7$

$$\text{nullity} = n - \mathrm{rank}(A^T)$$
$$= 7 - 2 = 5$$

Addendum to Invertible Matrix Theorem

Theorem:

Let A be an $n \times n$ matrix. The following are equivalent to the statement that A is invertible.

(m) The columns of A form a basis for $\mathbb{R}^n$

(n) $\operatorname{Col} A = \mathbb{R}^n$

(o) $\dim \operatorname{Col} A = n$

(p) $\operatorname{rank} A = n$

(q) $\operatorname{Nul} A = \{\mathbf{0}\}$

(r) $\dim \operatorname{Nul} A = 0$

Section 5.1: Eigenvectors and Eigenvalues

Consider the matrix A and vectors $\mathbf{u}$ and $\mathbf{v}$.

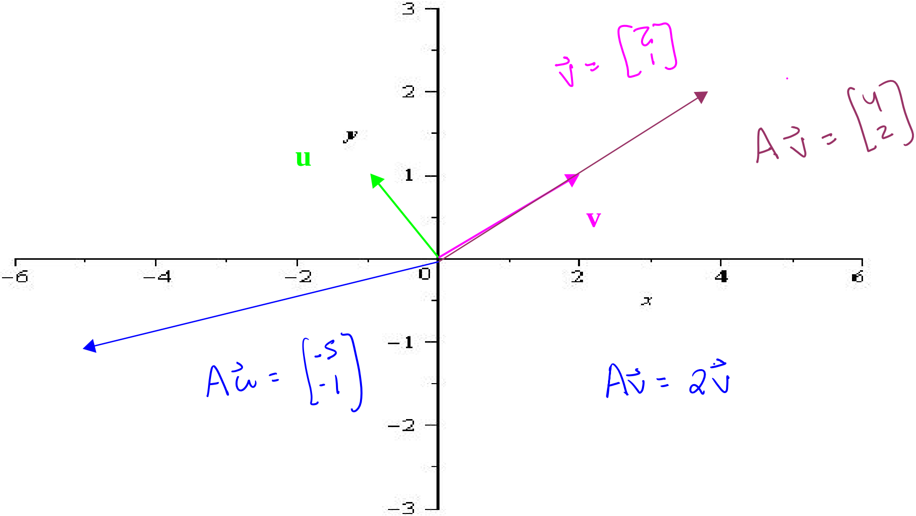
$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Plot $\mathbf{u}$, $A\mathbf{u}$, $\mathbf{v}$, and $A\mathbf{v}$ on the axis on the next slide.

$$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Example Plot



Figure

Eigenvalues and Eigenvectors

Remark: Note the action of A on the two vectors seems fundamentally different.

- A seems to both stretch and rotate the vector $\mathbf{u}$.
- The *action of A* on the vector $\mathbf{v}$ is just a stretch/compress.

$$\boxed{A\mathbf{v} \text{ is in } \mathrm{Span}\{\mathbf{v}\}.}$$

We wish to consider matrices with vectors that satisfy relationships such as

$$A\mathbf{x} = 2\mathbf{x}, \quad \text{or} \quad A\mathbf{x} = -4\mathbf{x}, \quad \text{or more generally} \quad A\mathbf{x} = \lambda\mathbf{x}$$

for constant λ—and for nonzero vector $\mathbf{x}$.

Definition of Eigenvector and Eigenvalue

Definition:

Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ is called an **eigenvector** of the matrix A.

A scalar λ such that there exists a nonzero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is called an **eigenvalue** of the matrix A. Such a nonzero vector $\mathbf{x}$ is an *eigenvector corresponding to* λ.

Note that built right into this definition is that the eigenvector $\mathbf{x}$ <u>MUST BE</u> a nonzero vector!

Example

The number $\lambda = -4$ is an eigenvalue of the matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Find the corresponding eigenvectors.

we seek $\vec{x}$ in $\mathbb{R}^2$ such that $A\vec{x} = -4\vec{x}$.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$A\vec{x} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 6x_2 \\ 5x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} -4x_1 \\ -4x_2 \end{bmatrix}$$

$$\begin{aligned} x_1 + 6x_2 &= -4x_1 \\ 5x_1 + 2x_2 &= -4x_2 \end{aligned} \implies \begin{aligned} 5x_1 + 6x_2 &= 0 \\ 5x_1 + 6x_2 &= 0 \end{aligned}$$

This equation is homogeneous with coefficient matrix $A - (-4)I$

$$\begin{bmatrix} 5 & 6 & 0 \\ 5 & 6 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 6/5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -\frac{6}{5} x_2$

x_2 - free

The eigen vectors look like

$$\vec{x} = x_2 \begin{bmatrix} -6/5 \\ 1 \end{bmatrix} \quad \text{for } x_2 \neq 0.$$

Let's check with one such vector.

Let $\vec{v} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$. This is taking $x_2 = 5$.

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} -6 \\ 5 \end{bmatrix} = \begin{bmatrix} 24 \\ -20 \end{bmatrix} = -4\begin{bmatrix} -6 \\ 5 \end{bmatrix}$$

Eigenspace

Definition:

Let A be an $n \times n$ matrix and λ and eigenvalue of A. The set of all eigenvectors corresponding to λ together with the zero vector—i.e. the set

$$\{\mathbf{x} \in \mathbb{R}^n \mid \text{ and } A\mathbf{x} = \lambda\mathbf{x}\},$$

is called the **eigenspace of A corresponding to λ**.

Remark: The eigenspace is the same as the null space of the matrix $A - \lambda I$. It follows that the eigenspace is a subspace of $\mathbb{R}^n$.

Example

The matrix $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ has eigenvalue $\lambda = 2$. Find a basis for the eigenspace of A corresponding to λ.

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

we're looking for solutions to

$$(A - 2I)\vec{x} = \vec{0}$$

$$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1/2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \frac{1}{2}x_2 - 3x_3$

x_2, x_3 are free

$$\vec{x} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the eigen space is

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Matrices with Nice Structure

Theorem:

If A is an $n \times n$ triangular matrix, then the eigenvalues of A are its diagonal elements.

Example: Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & \pi & 0 \\ -1 & 0 & 1 \end{bmatrix}$

They are $\lambda_1 = 3$, $\lambda_2 = \pi$, and

$\lambda_3 = 1$

Example

Suppose $\lambda = 0$ is an eigenvalue[1] of a matrix A. Argue that A is not invertible.

Since zero is an eigenvalue, there is a nonzero vector $\vec{x}$ such that $A\vec{x} = 0\vec{x} = \vec{0}$. That is, $A\vec{x} = \vec{0}$ has a nontrivial solution. By the invertible matrix theorem, A is not invertible.

[1] Eigenvectors must be nonzero vectors, but it is perfectly legitimate to have a zero eigenvalue!

Theorems

Theorem:

A square matrix A is invertible if and only if zero is **not** and eigenvalue.

Theorem:

If $\mathbf{v}_1, \dots, \mathbf{v}_r$ are eigenvectors of a matrix A corresponding to distinct eigenvalues, $\lambda_1, \dots, \lambda_r$, then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly independent.

Linear Independence

Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of a matrix A corresponding to distinct eigenvalues λ_1 and λ_2 (i.e.$\lambda_1 \neq \lambda_2$).

Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

Note $A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$

Consider $c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$

Let's create two equations.

E_1: multiply through by λ_1 assuming $\lambda_1 \neq 0$

$$c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_1 \vec{v}_2 = \lambda_1 \vec{0} = \vec{0}$$

Ex: multiply through by A

$$A\left(c_1\vec{v}_1 + c_2\vec{v}_2\right) = A\vec{0} = \vec{0}$$

$$c_1 A\vec{v}_1 + c_2 A\vec{v}_2 = \vec{0}$$

$$c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 = \vec{0}$$

$$c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_1 \vec{v}_2 = \vec{0}$$

subtract

$$c_2 \lambda_2 \vec{v}_2 - c_2 \lambda_1 \vec{v}_2 = \vec{0}$$

$$c_2 (\lambda_2 - \lambda_1) \vec{v}_2 = \vec{0}$$

$\vec{v}_2 \neq \vec{0}$ because it's an eigenvector.

$\lambda_2 - \lambda_1 \neq 0$ because $\lambda_1 \neq \lambda_2$

$\Rightarrow \quad c_2 = 0$.

The original equation becomes

$$c_1 \vec{v}_1 = \vec{0} \quad \Rightarrow \quad c_1 = 0$$

Hence $c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$ has only the trivial solution.

$\{\vec{v}_1, \vec{v}_2\}$ is lin. independent.

Section 5.2: The Characteristic Equation

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ by appealing to the fact that the equation $A\mathbf{x} = \lambda I_2\mathbf{x}$ can be restated as:

Find a nontrivial solution of the homogeneous equation

$$(A - \lambda I_2)\mathbf{x} = \mathbf{0}.$$

Consider $A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$$= \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

We can insist that $\det(A - \lambda I) = 0$.

This will guarantee nontrivial solutions to the homogeneous eqn.

$$\det(A-\lambda I) = (2-\lambda)(-6-\lambda) - 9$$

$$= \lambda^2 + 4\lambda - 12 - 9$$

$$= \lambda^2 + 4\lambda - 21$$

we want λ such that

$$\lambda^2 + 4\lambda - 21 = 0$$

$$(\lambda+7)(\lambda-3) = 0$$

We get two solutions $\lambda_1 = -7$ and

$\lambda_2 = 3$.

$$\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$$A-(-7)I = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix}$$

$$A-3I = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

Another Addendum to the Invertible Matrix Thm.

Theorem:

The $n \times n$ matrix A is invertible if and only if[a]

(s) The number 0 is not an eigenvalue of A.

(t) The determinant of A is nonzero.

[a] This is nothing new, we're just adding to the list.

Characteristic Equation

Definition:

For $n \times n$ matrix A, the expression $\det(A - \lambda I)$ is an n^{th} degree polynomial in λ. It is called the **characteristic polynomial** of A.

Definition:

The equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A.

Theorem:

The scalar λ is an eigenvalue of the matrix A if and only if it is a root of the characteristic equation.

Example

Find the characteristic equation for the matrix and identify all of its eigenvalues.

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

.

$$A - \lambda I = \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

The characteristic equation is

$$(5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda) = 0$$

$$(5-\lambda)^2(3-\lambda)(1-\lambda) = 0$$

The eigenvalues are

$$\lambda_1 = 5, \quad \lambda_2 = 3, \quad \lambda_3 = 1$$

Multiplicities

Definition:

The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation. The **geometric multiplicity** is the dimension of its corresponding eigenspace.

Example Find the algebraic and geometric multiplicity of the eigenvalue $\lambda = 5$ of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

.

The characteristic polynomial is

$$(5-\lambda)^2(3-\lambda)(1-\lambda)$$

The algebraic multiplicity is 2.

We need to consider a basis for the null space of $A - 5I$.

$$A - 5I = \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & -8 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$\xrightarrow{\text{rref}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1 is free

$x_2 = x_3 = x_4 = 0$

$\dim(\mathrm{Nul}(A - 5I)) = 1$ (one free variable)

The geometric multiplicity is

1.

Similarity

Definition:

Two $n \times n$ matrices A and B are said to be **similar** if there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

The mapping $A \mapsto P^{-1}AP$ is called a **similarity transformation**[a].

[a] **Note:** similarity is NOT related to row equivalence.

Theorem:

If A and B are similar matrices, then they have the same characteristic equation, and hence the same eigenvalues.

If $B = P^{-1}AP$, then $\det(B - \lambda I) = \det(A - \lambda I)$

$B - \lambda I = P^{-1}AP - \lambda I$ note $I = P^{-1} I P$

$= P^{-1}AP - \lambda P^{-1} I P$

$= P^{-1}(AP - \lambda I P)$

$= P^{-1}(A - \lambda I)P$

$\det(B - \lambda I) = \det\left(P^{-1}(A - \lambda I)P\right)$

$*\quad \det(XY) = \det(X)\det(Y)$

$$\det(B-\lambda I) = \det(P^{-1})\det(A-\lambda I)\det(P)$$

scalar product

$$= \det(A-\lambda I)\underbrace{\det(P^{-1})\det(P)}_{=1}$$

$$= \det(A-\lambda I)$$

$$\det(B-\lambda I) = \det(A-\lambda I)$$