July 13 Math 3260 sec. 51 Summer 2023

Section 5.2: The Characteristic Equation

Definition: Eigenvalues & Eigenvectors

Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ is called an **eigenvector** of the matrix A.

A scalar λ such that there exists a nonzero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is called an **eigenvalue** of the matrix A. Such a nonzero vector $\mathbf{x}$ is an *eigenvector corresponding to* λ.

Eigenspace

Definition:

Let A be an $n \times n$ matrix and λ and eigenvalue of A. The set of all eigenvectors corresponding to λ together with the zero vector—i.e. the set

$$\{\mathbf{x} \in \mathbb{R}^n \mid \text{ and } A\mathbf{x} = \lambda\mathbf{x}\},$$

is called the **eigenspace of A corresponding to λ**.

Remark: The eigenspace is the same as the null space of the matrix $A - \lambda I$. It follows that the eigenspace is a subspace of $\mathbb{R}^n$.

Characteristic Equation

Definition:

For $n \times n$ matrix A, the expression $\det(A - \lambda I)$ is an n^{th} degree polynomial in λ. It is called the **characteristic polynomial** of A.

Definition:

The equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A.

Theorem: Eigenvalues

The scalar λ is an eigenvalue of the matrix A if and only if it is a root of the characteristic equation.

Definition: Multiplicities

The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation. The **geometric multiplicity** is the dimension of its corresponding eigenspace.

For an eigenvalue λ_i the

- **algebraic mutliplicity** = the power on $(\lambda - \lambda_i)$ in $\det(A - \lambda I) = 0$
- **geometric multiplicity** = number of free variables in $(A - \lambda_i I)\mathbf{x} = \mathbf{0}$.

Similarity

Definition: Similar Matrices

Two $n \times n$ matrices A and B are said to be **similar** if there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

The mapping $A \mapsto P^{-1}AP$ is called a **similarity transformation**[a].

[a]**Note:** similarity is NOT related to row equivalence.

Theorem:

If A and B are similar matrices, then they have the same characteristic equation, and hence the same eigenvalues.

Example

Show that $A = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$ are similar with the matrix P for the similarity transformation given by $P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$.

We need to show that

$B = P^{-1}AP$

$\det(P) = 2-3 = -1$

$$P^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 6 & -12 \\ 3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$$

$$= B$$

Example Continued...

Show that the columns of P are eigenvectors of A where

$$A = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}.$$

Let $\vec{p}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{p}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

we need to show that $A\vec{p}_1 = \lambda_1 \vec{p}_1$

and $A\vec{p}_2 = \lambda_2 \vec{p}_2$

$$A\vec{p}_1 = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\vec{p}_1$$

$$A\vec{p}_2 = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ -4 \end{bmatrix} = -4\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Eigenvalues of a real matrix need not be real

Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 3 \\ -5 & 2 \end{bmatrix}$.

$$\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 3 \\ -5 & 2-\lambda \end{vmatrix}$$

$$= (4-\lambda)(2-\lambda)+15 = \lambda^2 - 6\lambda + 23$$

The characteristic equation is

$$\lambda^2 - 6\lambda + 23 = 0$$

Completing the square

$$\lambda^2 - 6\lambda + 9 + 14 = 0$$

$$(\lambda - 3)^2 = -14$$

$$\lambda = 3 \pm \sqrt{14}\, i$$

no real roots.

Section 5.3: Diagonalization

Motivational Example:

Determine the eigenvalues of the matrix D^3 (that's D cubed), where

$$D = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}.$$

$$D^2 = DD = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 3^2 & 0 \\ 0 & (-4)^2 \end{bmatrix}$$

$$D^3 = DD^2 = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} 3^2 & 0 \\ 0 & (-4)^2 \end{bmatrix} = \begin{bmatrix} 3^3 & 0 \\ 0 & (-4)^3 \end{bmatrix}$$

The eigen values of D^3 are

3^3 and $(-4)^3$

Recall that a matrix D is diagonal if it is both upper and lower triangular (its only nonzero entries are on the diagonal).

Theorem

If D is a diagonal matrix with diagonal entries d_{ii}, then D^k is diagonal with diagonal entries d_{ii}^k for positive integer k. Moreover, the eigenvalues of D are the diagonal entries.

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} \implies D^k = \begin{bmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{bmatrix}$$

Powers and Similarity

Suppose A and B are similar matrices with similarity transform matrix P—i.e., $B = P^{-1}AP$. Show that

a. A^2 and B^2 are similar with the same P,

b. A^3 and B^3 are similar with the same P.

$$B = P^{-1}AP$$

$$B^2 = (P^{-1}AP)^2$$

$$= (P^{-1}AP)(P^{-1}AP)$$

$$= P^{-1}A(PP^{-1})AP$$

$$= P^{-1}AIAP \qquad = P^{-1}AAP$$

$$B^2 = P^{-1} A^2 P$$

$$B^3 = B B^2 = (P^{-1} A P)(P^{-1} A^2 P)$$

$$= P^{-1} A (P P^{-1}) A^2 P$$

$$= P^{-1} A I A^2 P$$

$$= P^{-1} A A^2 P$$

$$B^3 = P^{-1} A^3 P$$

Diagonalizability

Defintion:

An $n \times n$ matrix A is called **diagonalizable** if it is similar to a diagonal matrix D. That is, provided there exists a nonsingular matrix P such that $D = P^{-1}AP$—i.e. $A = PDP^{-1}$.

Theorem:

The $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In this case, the matrix P is the matrix whose columns are the n linearly independent eigenvectors of A.

Example

Diagonalize the matrix A if possible. $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

① Find the eigenvalues.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)\begin{vmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} - 3\begin{vmatrix} -3 & -3 \\ 3 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} -3 & -5-\lambda \\ 3 & 3 \end{vmatrix}$$

$$= (1-\lambda)\Big(-(5+\lambda)(1-\lambda) + 9\Big) - 3\left(-3+3\lambda + 9\right) + 3\left(-9 + 15 + 3\lambda\right)$$

$$= (1-\lambda)\left[-(-\lambda^2-4\lambda+5)+9\right]-3(3\lambda+6)+3(3\lambda+6)$$

$$= (1-\lambda)(\lambda^2+4\lambda-5+9)$$

$$= (1-\lambda)(\lambda^2+4\lambda+4) = (1-\lambda)(\lambda+2)^2$$

Characteristic equation

$$(1-\lambda)(\lambda+2)^2 = 0 \Rightarrow \lambda_1 = 1, \quad \lambda_2 = -2$$

Find eigenvectors:

$\lambda_1 = 1$ $\quad A - 1I = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = x_3$, $x_2 = -x_3$, x_3 free

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

$$\lambda_2 = -2 \quad A - (-2)I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2 - x_3$

x_2, x_3 - free

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We have 3 lin. ind eigenvectors. A is diagonalizable.

A matrix $P = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = D$$

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$$

$$\lambda_1 \quad \lambda_2 \quad \lambda_2$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

Example

Diagonalize the matrix A if possible. $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. (With a little effort, it can be shown that the characteristic polynomial of A is $(1-\lambda)(2+\lambda)^2$.)

Characteristic eqn $\quad (1-\lambda)(2+\lambda)^2 = 0$

$\lambda_1 = 1, \quad \lambda_2 = -2$

Find eigenvectors

$$A - 1I = \begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$A - (-2)I = \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$x_1 = -x_2$

x_2 - free

$x_3 = 0$

we get one basis element for the eigenspace.

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

A doesn't have 3 lin. ind. eigenvectors. A is <u>not</u> diagonalizable.

Sufficient Condition for Diagonalizability

Recall: Eigenvectors corresponding to different eigenvalues are linearly independent.

Theorem:

If the $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Remark

This is a *sufficiency* condition, not a *necessity* condition. This means that if a matrix has n different eigenvalues, it's guaranteed to be diagonalizable. If it has repeated eigenvalues, it may or may not be diagonalizable.

More on Diagonalizability

Theorem:

Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

(a) The geometric multiplicity of λ_k is less than or equal to the algebraic multiplicity of λ_k.

(b) The matrix is diagonalizable if and only if the sum of the geometric multiplicities is n.

(c) If A is diagonalizable, and $\mathcal{B}_k$ is a basis for the eigenspace for λ_k, then the collection (union) of bases $\mathcal{B}_1, \ldots, \mathcal{B}_p$ is a basis for $\mathbb{R}^n$.

Remark: The union of the bases referred to in part (c) is called an **eigenvector basis** of $\mathbb{R}^n$ for the matrix A.

Example

Diagonalize the matrix if possible. $A = \begin{bmatrix} 8 & -6 \\ 9 & -7 \end{bmatrix}$.

Find e.values

$$A - \lambda I = \begin{bmatrix} 8-\lambda & -6 \\ 9 & -7-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (8-\lambda)(-7-\lambda) + 54$$

$$= \lambda^2 - \lambda - 56 + 54$$

$$= \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$$

$(\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda_1 = 2$ and $\lambda_2 = -1$

For $\lambda_1 = 2$ $\quad A - 2I = \begin{bmatrix} 6 & -6 \\ 9 & -9 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 = x_2$, x_2 free

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -1$ $\quad A - (-1)I = \begin{bmatrix} 9 & -6 \\ 9 & -6 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -2/3 \\ 0 & 0 \end{bmatrix}$

$$x_1 = \frac{2}{3}x_2 \qquad x_2 \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

a choice $\quad \vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

A matrix P is $\quad P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

givens $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

Example Continued...

Find A^3 where $A = \begin{bmatrix} 8 & -6 \\ 9 & -7 \end{bmatrix}$.

$D = P^{-1}AP$

$D^3 = P^{-1}A^3P$

$A^3 = PD^3P^{-1}$

$D^3 = \begin{bmatrix} 2^3 & 0 \\ 0 & (-1)^3 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

$A^3 = PD^3P^{-1}$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 24 & -16 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & -18 \\ 27 & -19 \end{bmatrix}$$

Section 6.1: Inner Product, Length, and Orthogonality

Definition

For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ we define the **inner product** of $\mathbf{u}$ and $\mathbf{v}$ (also called the **dot product**) by the **matrix product**

$$\mathbf{u}^T\mathbf{v} = [u_1\ u_2\ \cdots\ u_n]\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Remark: Note that this product produces a scalar. It is sometimes called a *scalar product*.

Remark: This is the same dot product we defined before (without using the transpose) that we used to compute a matrix product entries.

The Norm

Definition

The **norm** of the vector $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ is the nonnegative number

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Theorem

For vector $\mathbf{v}$ in $\mathbb{R}^n$ and scalar c

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|.$$

Unit Vectors & Normalizing

Definition

A vector $\mathbf{u}$ in $\mathbb{R}^n$ for which $\|\mathbf{u}\| = 1$ is called a **unit vector.**

Remark

Given any nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$, we can find a unit vector in the direction of $\mathbf{v}$ by dividing $\mathbf{v}$ by its norm.

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} \quad \text{is a unit vector.}$$

This is called **normalizing** the vector.

Distance in $\mathbb{R}^n$

Definition:

For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the **distance between u and v** is denoted by

$$\text{dist}(\mathbf{u}, \mathbf{v}),$$

and is defined by

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Remark: This is the same as the traditional formula for distance used in $\mathbb{R}^2$ between points (x_0, y_0) and (x_1, y_1),

$$d = \sqrt{(y_1 - y_0)^2 + (x_1 - x_0)^2}.$$

Example

Find the distance between the vectors $\mathbf{u} = (4, 0, -1, 1)$ and $\mathbf{v} = (0, 0, 2, 7)$ in $\mathbb{R}^4$.

$$\vec{u} - \vec{v} = (4, 0, -3, -6)$$

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$

$$= \sqrt{4^2 + 0^2 + (-3)^2 + (-6)^2} = \sqrt{61}$$

Orthogonality

Definition:

Two vectors are $\mathbf{u}$ and $\mathbf{v}$ **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

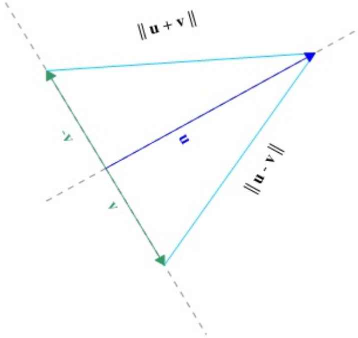

Figure: Note that two vectors are perpendicular if $\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{u} + \mathbf{v}\|$

Orthogonal and Perpendicular

Show that $\|\mathbf{u}-\mathbf{v}\| = \|\mathbf{u}+\mathbf{v}\|$ if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

Let's show that $\|\vec{u}-\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$ if and only if $\vec{u}\cdot\vec{v} = 0$. Note

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

$$= (\vec{u}-\vec{v})^T(\vec{u}-\vec{v})$$

$$= \vec{u}^T\vec{u} - \vec{v}^T\vec{u} - \vec{u}^T\vec{v} + \vec{v}^T\vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v}$$

$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u}\cdot\vec{v}$$

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v}$$

If $\|\vec{u}-\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$, then

$$4\vec{u}\cdot\vec{v} = 0 \implies \vec{u}\cdot\vec{v} = 0$$

If $\vec{u}\cdot\vec{v} = 0$, then

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$$

The Pythagorean Theorem

Theorem:

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

This follows immediately from the observation that

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v}.$$

The two vectors are defined as being orthogonal precisely when $\mathbf{u}\cdot\mathbf{v} = 0$.