July 14 Math 3260 sec. 51 Summer 2025

5.5 Compositions & Similarity

Suppose $S : R^n \to R^p$ and $T : R^p \to R^m$ are linear transformations, then we can ask about the composition

$$T \circ S : R^n \to R^m.$$

$$(T \circ S)(\vec{x}) = T(S(\vec{x})) = T(A_S \vec{x}) = A_T(A_S \vec{x})$$

$$= (A_T A_S)\vec{x}$$

Suppose

$$S(\vec{x}) = A_S\vec{x}, \quad \text{and} \quad T(\vec{y}) = A_T\vec{y}.$$

How is the standard matrix for the composition related to the standard matrices of S and T? It's the product $A_T A_S$

This gives the primary motivation for the way matrix multiplication is defined.

Matrix Multiplication is Composition

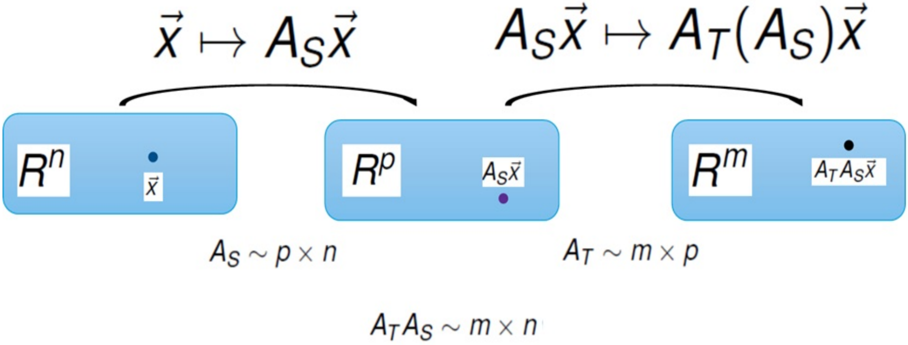

Figure: $\vec{x}$ is mapped from R^n to R^p, then $A_S\vec{x}$ is mapped from R^p to R^m. The composition maps from R^n to R^m.

$$\begin{aligned} S : R^n \longrightarrow R^p \quad &\Longrightarrow \quad A_S \sim p \times n \\ T : R^p \longrightarrow R^m \quad &\Longrightarrow \quad A_T \sim m \times p \\ T \circ S : R^n \longrightarrow R^m \quad &\Longrightarrow \quad A_TA_S \sim m \times n \end{aligned}$$

Example

Suppose that $S : R^3 \to R^2$ is the linear transformation

$$S(\langle x_1, x_2, x_3 \rangle) = \langle 2x_1 + x_2, 2x_1 + x_2 + x_3 \rangle$$

and suppose that $T : R^2 \to R^3$ is the linear transformation

$$T(\langle x_1, x_2 \rangle) = \langle -x_1, 3x_1 - x_2, -2x_1 + 3x_2 \rangle .$$

Find the standard matrix for the composition $T \circ S$.

The final matrix would be 3x3.

Find A_S and A_T

$S(\vec{e}_1) = \langle 2, 2 \rangle$

$S(\vec{e}_2) = \langle 1, 1 \rangle$

$S(\vec{e}_3) = \langle 0, 1 \rangle$

$$A_S = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

$T(\vec{e}_1) = \langle -1, 3, -2 \rangle$

$T(\vec{e}_2) = \langle 0, -1, 3 \rangle$

$$A_T = \begin{bmatrix} -1 & 0 \\ 3 & -1 \\ -2 & 3 \end{bmatrix}$$

For $T \circ S$ the matrix $A_{T \circ S} = A_T A_S$.

$$A_{T \circ S} = A_T A_S = \begin{bmatrix} -1 & 0 \\ 3 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 & 0 \\ 4 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

Reflection in Line Through the Origin

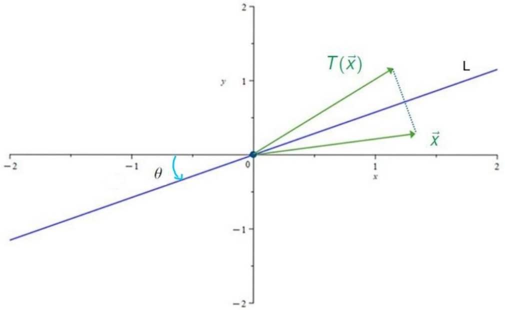

Figure: We want a transformation T to reflect a vector through a line through the origin that makes an angle θ with the x_1-axis.

We'll do this in three steps.

Start w/ Line & Vector

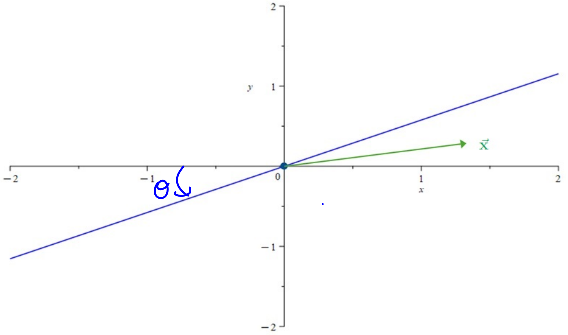

Figure: The line L makes an angle θ with respect to the x_1-axis. We want to reflect the vector $\vec{x}$ through it.

Apply: $R_{-\theta}(\vec{x}) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \vec{x}$, call this $\vec{y}$, i.e., $\vec{y} = R_{-\theta}(\vec{x})$

Rotate θ Clockwise

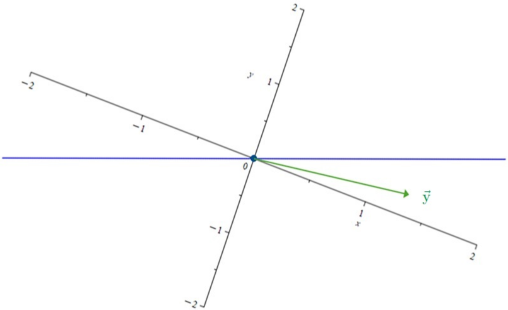

Figure: Rotate through θ clockwise using $R_{-\theta}$. L becomes the x_1-axis.

Next apply: $P_{x_1}(\vec{y}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \vec{y}$, call this $\vec{z}$, i.e., $\vec{z} = P_{x_1}(\vec{y})$

Reflect Through x_1-axis

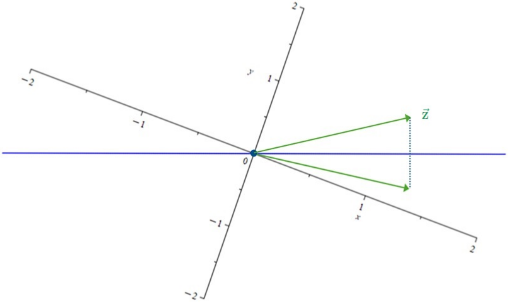

Figure: Reflect through the x_1-axis using the Reflection transformation P_{x_1}.

Finally apply: $R_\theta(\vec{z}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{z}$, this is $T(\vec{x})$, i.e., $T(\vec{x}) = R_\theta(\vec{z})$

Rotate θ Counter Clockwise

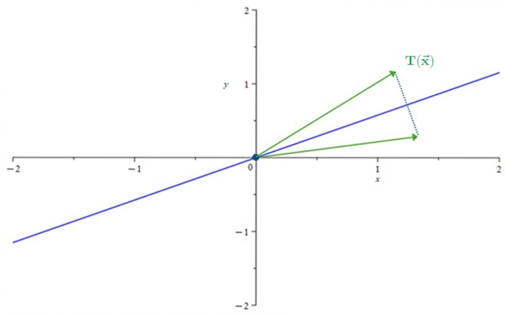

Figure: Then we rotation back through θ in the counterclockwise direction by applying the transformation R_θ.

The total transformation is

$$T = R_\theta \circ P_{x_1} \circ R_{-\theta}, \qquad \text{(recall } R_\theta = R_{-\theta}^{-1}\text{)}$$

Similarity

Our complicated reflection through a line that was not horizontal can be done with the “simple” reflection through a horizontal line. Note that the matrix for this is the product

$$A_T = A_{-\theta}^{-1} A_{P_{x_1}} A_{-\theta}.$$

Note that the form of this is a matrix sandwiched between a matrix and its inverse. The complicated ~~projection~~ reflection T is said to be **similar** to the simple ~~projection~~ reflection P_{x_1}.

Note that this only makes sense if we're mapping from R^n back to itself.

Similarity

A linear transformation $T : R^n \to R^n$ is said to be **similar** to a linear transformation $S : R^n \to R^n$ if there exists an invertible linear transformation $P : R^n \to R^n$ such that

$$T = P^{-1} \circ S \circ P.$$

Likewise, an $n \times n$ matrix A is said to be **similar** to an $n \times n$ matrix B, if there exists an invertible $n \times n$ matrix C such that

$$A = C^{-1}BC.$$

Note that this can be viewed either direction since $T = P^{-1} \circ S \circ P$ and $A = C^{-1}BC$ imply

$$S = P \circ T \circ P^{-1} \quad \text{and} \quad B = CAC^{-1}$$

Using Similarity

Consider the matrix $A = \begin{bmatrix} -8 & -3 \\ 18 & 7 \end{bmatrix}$. Suppose we want to compute A^9.

$$A^9 = AAAAAAAAA = \underbrace{\begin{bmatrix} -8 & -3 \\ 18 & 7 \end{bmatrix}\begin{bmatrix} -8 & -3 \\ 18 & 7 \end{bmatrix}\cdots\begin{bmatrix} -8 & -3 \\ 18 & 7 \end{bmatrix}}_{\text{nine factors of } A.}$$

Compare that to computing D^9 if $D = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$.

$$D^2 = DD = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (-2)^2 & 0 \\ 0 & 1^2 \end{bmatrix}$$

$$D^3 = D^2 D = \begin{bmatrix} (-2)^2 & 0 \\ 0 & 1^2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (-2)^3 & 0 \\ 0 & 1^3 \end{bmatrix}$$

$\vdots$

$$D^9 = \begin{bmatrix} (-2)^9 & 0 \\ 0 & 1^9 \end{bmatrix}$$

$$A = \begin{bmatrix} -8 & -3 \\ 18 & 7 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \quad C^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

What if we know that $D = C^{-1}AC$ which means that $A = CDC^{-1}$?

Show that $D^2 = C^{-1}A^2C$ and $D^3 = C^{-1}A^3C$.

$$D^2 = DD = (C^{-1}AC)(C^{-1}AC) = C^{-1}A(CC^{-1})AC$$

$$= C^{-1}AIAC = C^{-1}AAC = C^{-1}A^2C$$

$$D^3 = D^2D = (C^{-1}A^2C)(C^{-1}AC) = C^{-1}A^2(CC^{-1})AC$$

$$= C^{-1}A^2IAC = C^{-1}A^2AC = C^{-1}A^3C$$

Powers of Similar Matrices

If A and B are similar matrices, with $B = C^{-1}AC$ for some invertible matrix C, then for every integer $n \geq 1$

$$B^n = C^{-1}A^nC.$$

This means that $A^9 = CD^9C^{-1}$. That's two matrix multiplications instead of eight matrix multiplications.

$$A = \begin{bmatrix} -8 & -3 \\ 18 & 7 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \quad C^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$A^9 = C D^9 C^{-1} \qquad D^9 = \begin{bmatrix} (-2)^9 & 0 \\ 0 & 1^9 \end{bmatrix} = \begin{bmatrix} -512 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^9 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -512 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -512 & 512 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(-512)-2 & 3(512)+3 \\ 2(-512)-2 & 2(512)+3 \end{bmatrix} = \begin{bmatrix} -1538 & 1539 \\ -1026 & 1027 \end{bmatrix}$$

5.6 Linear Transformations for General Vector Spaces

Linear Transformation

Suppose V and W are vector spaces. A **linear transformation** from V to W is a function $T : V \to W$ such that for each pair of vectors $\vec{x}$ and $\vec{y}$ in V and for any scalar c

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$, and
2. $T(c\vec{x}) = cT(\vec{x})$.

The only difference is that we've replaced R^n and R^m with V and W.

Example

Consider the vector spaces $C^1(R)$ and $C^0(R)$. The transformation

$$D : C^1(R) \to C^0(R)$$

defined by

$$D(f) = f'$$

is a linear transformation.

Recall $(f+g)' = f' + g'$

$$D(f+g) = D(f) + D(g)$$

$$(cf)' = cf'$$

$$D(cf) = c\,D(f)$$

Theorem

If V and W are vector spaces and $T : V \to W$ is a linear transformation, then $T(\vec{0}_V) = \vec{0}_W$.

Proof: Let $\vec{v}$ be any vector V, and let $\vec{w} = T(\vec{v})$.

So $\vec{w}$ is some vector in W. Recall

$0\vec{v} = \vec{0}_V$ and $0\vec{w} = \vec{0}_W$.

$$T(\vec{0}_V) = T(0\vec{v}) = 0T(\vec{v}) = 0\vec{w} = \vec{0}_W.$$

■

Range & Kernel

If V and W are vector spaces and $T : V \to W$ is a linear transformation, we define the **range** of T to be

$$\mathrm{range}\,(T) = \{\vec{y} \in W \mid T\,(\vec{x}) = \vec{y} \text{ for at least one } \vec{x} \in V\}$$

and we define the **kernel** (also called **null space**) of T to be

$$\ker\,(T) = \left\{\vec{x} \in V \mid T\,(\vec{x}) = \vec{0}_W\right\}.$$

Theorem

Let V and W be vector spaces and $T : V \to W$ be a linear transformation. Then $\mathrm{range}(T)$ is a subspace of W and $\ker(T)$ is a subspace of V.

$$\dim(\mathrm{range}(T)) + \dim(\ker(T)) = \dim(V).$$

Invertibility

Suppose that V and W are vector spaces and suppose that $T : V \to W$ is a linear transformation. We say that T is **invertible** if $\text{range}(T) = W$ and T is also one–to–one. If T is invertible, then the **inverse** of T is defined to be the function $T^{-1} : W \to V$ defined by

$$T^{-1}(\vec{y}) = \vec{x} \text{ where } \vec{x} \text{ is the unique vector in } V \text{ such that } T(\vec{x}) = \vec{y}.$$

The equation $\dim(\text{range}(T)) + \dim(\ker(T)) = \dim(V)$ implies that T can only be invertible if $\dim(V) = \dim(W)$. Of course, even if $\dim(V) = \dim(W)$ a transformation may not be invertible.

It is also the case that if T is invertible, then $T^{-1} : W \to V$ is also a linear transformation.

Example $D : C^1(R) \to C^0(R)$ where $D(f) = f'$.

Let's show that D is not invertible[1].

1. If $f(x) = \cos(x)$, find $D(f)$ $= -\sin x$

2. If $g(x) = \cos(x) - 3$, find $D(g)$ $= -\sin x$

3. How many solutions are there to the equation $D(f) = -\sin(x)$?

   Infinitely many f can be $f(x) = \cos x + C$ for any scalar C.

4. Why does the above imply that D is not invertible?

   D is not one to one.

5. What is $\ker(D)$? $f \in \ker(D)$ if $D(f) = Z(x) = 0$

   The kernel contains all constant functions

[1]The Fundamental Theorem of Calculus does indicate that $\mathrm{range}(D) = C^0(R)$.

Powers of $T : V \to V$

If $T : V \to V$ is a linear transformation, we can compose T with itself and represent such compositions as "power."

$$\begin{aligned} T^2 &= T \circ T \\ T^3 &= T \circ T \circ T \\ &\vdots \end{aligned}$$

For example, consider $D : C^\infty(R) \to C^\infty(R)$ defined by $D(f) = f'$. Then

$$D^2(f) = f'', \quad D^3(f) = f''', \quad \ldots \quad D^n(f) = f^{(n)}.$$

Remember this Lemma?

Lemma

Suppose that S is a subspace of a vector space V and $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is an ordered basis of S. If $\vec{x}$ and $\vec{y}$ are any two vectors in S and c is any scalar then

1. $[\vec{x} + \vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$ and
2. $[c\vec{x}]_{\mathcal{B}} = c\,[\vec{x}]_{\mathcal{B}}$.

This means that the transformation from S to R^k that maps $\vec{x}$ in S to the coordinate vector $[\vec{x}]_{\mathcal{B}}$ in R^k is a **linear transformation**!

Plus, $\dim(S) = k$ because there are k basis elements, and $\dim(R^k) = k$, so the dimensions of the domain and codomain are the same, k.

Finite Dimensional Subspaces & Coordinate Mappings

Suppose S is a finite dimensional subspace of a vector space V and $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is an ordered basis of S. Recall that for vector $\vec{x} \in S$, if

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k,$$

the the coordinate vector relative to the basis $\mathcal{B}$ is

$$[\vec{x}]_{\mathcal{B}} = \langle c_1, c_2, \dots, c_k\rangle.$$

The mapping from $\vec{x}$ to $[\vec{x}]_{\mathcal{B}}$ is a linear transformation. To refer to the transformation to go back from $[\vec{x}]_{\mathcal{B}}$ to $\vec{x}$, we'll write

$$\left[[\vec{x}]_{\mathcal{B}}\right]^{-1} = \vec{x}, \quad \text{that is} \quad \left[\langle c_1, c_2, \dots, c_k\rangle\right]_{\mathcal{B}}^{-1} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k.$$

We're going to call this process, $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$, of going from $\vec{x}$ to $[\vec{x}]_{\mathcal{B}}$ a **coordinate mapping**.

Examples

Consider a simple example. Let $S = \mathbb{P}_2$ with ordered basis $\mathcal{B} = \{1, x, x^2\}$. Find

1. $[p]_{\mathcal{B}}$ if $p(x) = 2x^2 - 4x + 5$

   $[p]_{\mathcal{B}} = \langle c_1, c_2, c_3 \rangle$ if $p(x) = c_1(1) + c_2 x + c_3 x^2$

   $[p]_{\mathcal{B}} = \langle 5, -4, 2 \rangle$

2. $\left[\langle -4, 3, 12 \rangle\right]_{\mathcal{B}}^{-1} = -4(1) + 3x + 12x^2 = -4 + 3x + 12x^2$

3. $\left[6x - 3x^2 + 19\right]_{\mathcal{B}} = \langle 19, 6, -3 \rangle$

Examples

Consider a slightly more complicated example. Let $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \,\middle|\, a, d \in R \right\}$. This is the subspace of $M_{2\times 2}$ matrices all having zero off the main diagonal. Consider the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Find

$$\left[\begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \right]_{\mathcal{B}}$$

$= \langle 5, 2 \rangle$

$\left[\begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \right]_{\mathcal{B}} = \langle c_1, c_2 \rangle$ where

$\begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} c_1 + c_2 & 0 \\ 0 & c_2 \end{bmatrix}$

$c_2 = 2$ $\quad c_1 = 7 - c_2 = 5$

Examples

$$S = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \,\middle|\, a, d \in R \right\}, \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Find

$[\langle 4, -1 \rangle]_{\mathcal{B}}^{-1}$ $= 4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

Working with Coordinate Vectors

Let S be a finite dimensional subspace of some vector space V, and let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be an ordered basis for S.

Goal: We want to understand a linear transformation $T : S \to S$.

Process:

- Pass to coordinate vectors in R^k using the coordinate mapping $[\cdot]_{\mathcal{B}}$.
- Find a transformation $T_{\mathcal{B}} : R^k \to R^k$ that does *what T does in R^k*. This means we have to find the right matrix $A_{\mathcal{B}}$.
- Do the transformation $T_{\mathcal{B}} : R^k \to R^k$ on vectors in R^k using matrix multiplication.
- Pass from coordinate vectors back to the images under T (the vectors in S) using the inverse of the coordinate mapping $[\cdot]_{\mathcal{B}}^{-1}$.

Note: The dot in $[\cdot]_{\mathcal{B}}$ is just a place holder. It just means *something goes there*.

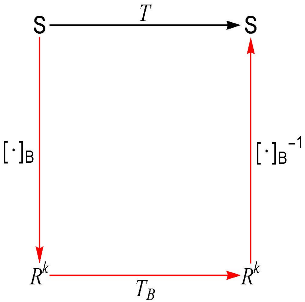

Figure: Schematic for construction of $T = [\cdot]_{\mathcal{B}}^{-1} \circ T_{\mathcal{B}} \circ [\cdot]_{\mathcal{B}}$.

$T_{\mathcal{B}}$ will have standard matrix $A_{\mathcal{B}}$ which we can call the **matrix of the linear transformation T with respect to the ordered basis $\mathcal{B}$.**

To make the notation less complicated, we'll drop the $\mathcal{B}$ as long as the context is clear. That is we'll write

$$[\cdot] \text{ in place of } [\cdot]_{\mathcal{B}}, \quad \text{and } A \text{ instead of } A_{\mathcal{B}}.$$

The Matrix A for $T_{\mathcal{B}} : R^k \to R^k$

For $\vec{x} \in S$

$$T(\vec{x}) = \left[A[\vec{x}]\right]^{-1}$$

where A is the $k \times k$ matrix whose columns are the coordinate vectors of the images of the basis elements under T.

$$\text{Col}_j(A) = \left[T(\vec{v}_j)\right]. \qquad \text{(the } \vec{v}_j\text{'s are the basis vectors)}$$

1. Put the basis elements from $\mathcal{B}$ into T.
2. Write their coordinate vectors relative to $\mathcal{B}$.
3. Make these the columns of a matrix A.
4. Do whatever we want with A.

Theorem

Let S be a finite dimensional subspace of a vector space V and $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be an ordered basis for S. Let $T : S \to S$ be a linear transformation and let A be the matrix of T with respect[a] to the ordered basis $\mathcal{B}$. Then

1. For any vector $\vec{y} \in \operatorname{range}(T)$, $T(\vec{x}) = \vec{y}$ if and only if $A[\vec{x}] = [\vec{y}]$.

2. The set of vectors $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_p\}$ is a basis for $\operatorname{range}(T)$ if and only if the set of vectors $\{[\vec{y}_1], [\vec{y}_2], \ldots, [\vec{y}_p]\}$ is a basis for $\mathcal{CS}(A)$.

3. The set of vectors $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_q\}$ is a basis for $\ker(T)$ if and only if the set of vectors $\{[\vec{x}_1], [\vec{x}_2], \ldots, [\vec{x}_q]\}$ is a basis for $\mathcal{N}(A)$.

4. For any integer $n \geq 1$ and any $\vec{x} \in S$, $T^n(\vec{x}) = [A^n[\vec{x}]]^{-1}$.

[a] $\mathrm{Col}_j(A) = \left[T(\vec{v}_j)\right]$.

Example: $\mathbb{P}_3 = \{p_0 + p_1 x + p_2 x^2 + p_3 x^3 \mid p_0, p_1, p_2, p_3 \in R\}$

Let $D : \mathbb{P}_3 \to \mathbb{P}_3$, with $D(f) = f'$. We can use the ordered basis

$$\mathcal{B} = \{1, x, x^2, x^3\}.$$

Identify the matrix A with respect to the basis $\mathcal{B}$ for D.

We need $D(\text{basis elements})$

$D(1) = 0$

$D(x) = 1$

$D(x^2) = 2x$

$D(x^3) = 3x^2$

need $[D(\text{basis elements})]$

$[D(1)] = [0] = \langle c_0, c_1, c_2, c_3 \rangle$ where

$0 = c_0(1) + c_1 x + c_2 x^2 + c_3 x^3$

$[D(1)] = \langle 0, 0, 0, 0 \rangle$

$D : \mathbb{P}_3 \to \mathbb{P}_3$, with $D(f) = f'$ $\quad \mathcal{B} = \{1, x, x^2, x^3\}$.

$$[D(x)] = [1] = \langle 1, 0, 0, 0 \rangle$$

$$[D(x^2)] = [2x] = \langle 0, 2, 0, 0 \rangle$$

$$[D(x^3)] = [3x^2] = \langle 0, 0, 3, 0 \rangle$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$D : \mathbb{P}_3 \to \mathbb{P}_3$, with $D(f) = f'$ $\quad \mathcal{B} = \{1, x, x^2, x^3\}$, $\quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Let $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$. Find $D(p)$ by

1. finding $[p]$ $= \langle p_0, p_1, p_2, p_3 \rangle$
2. then finding $A[p]$.

$$A[p] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \langle p_0, p_1, p_2, p_3 \rangle = \langle p_1, 2p_2, 3p_3, 0 \rangle$$

3. then finding $D(p) = \left[A[p]\right]^{-1}$.

$$D(p) = \left[\langle p_1, 2p_2, 3p_3, 0 \rangle\right]^{-1} = p_1(1) + 2p_2 x + 3p_3 x^2 + 0x^3$$

$$= p_1 + 2p_2 x + 3p_3 x^2$$

$D : \mathbb{P}_3 \to \mathbb{P}_3$, with $D(f) = f'$ $\quad \mathcal{B} = \{1, x, x^2, x^3\}$ $\quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Identify $\mathrm{range}(D)$ and $\ker(D)$.

For the range, get pivot columns of A.

$\mathrm{rref}(A) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

pivot columns are 2^{nd}, 3^{rd}, and 4^{th}

A basis for $CS(A)$ is

$\{ \langle 1,0,0,0\rangle, \langle 0,2,0,0\rangle, \langle 0,0,3,0\rangle \}$.

we need inverses

$[\langle 1,0,0,0\rangle]^{-1} = 1$, $\quad [\langle 0,2,0,0\rangle]^{-1} = 2x$

1. $\mathrm{range}(T)$ has basis $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_p\}$ if $\mathcal{CS}(A)$ has basis $\{[\vec{y}_1], [\vec{y}_2], \ldots, [\vec{y}_p]\}$
2. $\ker(T)$ has basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_q\}$ if $\mathcal{N}(A)$ has basis $\{[\vec{x}_1], [\vec{x}_2], \ldots, [\vec{x}_q]\}$

$D : \mathbb{P}_3 \to \mathbb{P}_3$, with $D(f) = f'$ $\quad \mathcal{B} = \{1, x, x^2, x^3\}$ $\quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$[\langle 0,0,3,0\rangle]^{-1} = 3x^2$

A basis for range(D) is $\{1, 2x, 3x^2\}$.

For $\ker(D)$ find a basis for $N(A)$.

$A\vec{x} = \vec{0}$. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ if $\vec{x} = \langle x_1, x_2, x_3, x_4\rangle$

$x_2 = x_3 = x_4 = 0$

x_1 is free.

$\vec{x} = \langle x_1, 0, 0, 0\rangle = x_1\langle 1,0,0,0\rangle$ a basis for $N(A)$

is $\{\langle 1,0,0,0\rangle\}$.

A basis for $\ker(D)$ is $\{1\}$.

$D : \mathbb{P}_3 \to \mathbb{P}_3$, with $D(f) = f'$ $\quad \mathcal{B} = \{1, x, x^2, x^3\}$ $\quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Find the vectors p in $\mathbb{P}_3$ such that $D(p) = 2x^2 - 3x + 4$. In the language of calculus, this is the same as finding all solutions to $\int (2x^2 - 3x + 4)\, dx$.

Let $q(x) = 2x^2 - 3x + 4$, $[q] = \langle 4, -3, 2, 0 \rangle$

Solve $A[p] = [q]$ Use $[A \mid [q]]$

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 2 & 0 & -3 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -3/2 \\ 0 & 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$p_1 = 4$

$p_2 = \frac{-3}{2}$

$p_3 = \frac{2}{3}$

p_0 is free

If $[p] = \langle p_0, p_1, p_2, p_3 \rangle$

$= \langle p_0, 4, -3/2, 2/3 \rangle$

$p(x) = \left[\langle p_0, 4, -\frac{3}{2}, \frac{2}{3} \rangle\right]^{-1} = p_0 + 4x - \frac{3}{2}x^2 + \frac{2}{3}x^3$

$p(x) = 4x - \frac{3}{2}x^2 + \frac{2}{3}x^3 + C$

Example

Let $S = \mathrm{Span}\{e^{2x}, xe^{2x}\}$ be the subspace of $C^\infty(R)$ with ordered basis $\mathcal{B} = \{e^{2x}, xe^{2x}\}$, and let $D : S \to S$ be the derivative transformation $D(f) = f'$. Find the matrix A with respect to the basis, and use it to evaluate

$$\int xe^{2x}\,dx.$$

That is, find all vectors f in S such that $D(f) = xe^{2x}$.

we need $D(e^{2x})$ and $D(xe^{2x})$

$D(e^{2x}) = 2e^{2x}$ $\qquad [D(e^{2x})] = \langle 2, 0\rangle$

$D(xe^{2x}) = e^{2x} + 2xe^{2x}$ $\qquad [D(xe^{2x})] = \langle 1, 2\rangle$

$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ we can find f

$A[f] = [g]$ where $g(x) = xe^{2x}$

$[g] = \langle 0, 1 \rangle$

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \langle c_1, c_2 \rangle = \langle 0, 1 \rangle$$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 2 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{2} \end{array}\right]$$

$[f] = \langle -\frac{1}{4}, \frac{1}{2} \rangle.$

$$f(x) = -\frac{1}{4} e^{2x} + \frac{1}{2} x e^{2x}$$

all vectors in S such that

$$D(f) = xe^{2x}$$

The constant of integration can't be picked up because constant functions are not elements of S.

$S = \mathrm{Span}\{e^{2x}, xe^{2x}\}, \quad D(f) = f' \quad A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

It's not too hard to find that $A^4 = \begin{bmatrix} 16 & 32 \\ 0 & 16 \end{bmatrix}$. Find $f^{(4)}(x)$ if $f(x) = 4e^{2x} - 3xe^{2x}$.

The coordinate vector for f is $[f] = \langle 4, -3 \rangle$.

$A^4[f] = \begin{bmatrix} 16 & 32 \\ 0 & 16 \end{bmatrix} \langle 4, -3 \rangle = \langle 64 - 96, -48 \rangle = \langle -32, -48 \rangle$

so $f^{(4)}(x) = \left[\langle -32, -48 \rangle\right]^{-1} = -32e^{2x} - 48xe^{2x}$

Turns out that $A^n = \begin{bmatrix} 2^n & n(2^{n-1}) \\ 0 & 2^n \end{bmatrix}$.