July 16 Math 3260 sec. 51 Summer 2025

5.7 Isomorphism of Vector Spaces

We were able to exploit R^4 to do operations in $\mathbb{P}_3$. Somehow, these spaces have the same structure. There's a name for this. It's called being

isomorphic.

Isomorphic

A vector space V is said to be **isomorphic** to a vector space W if there exists an invertible linear transformation $T : V \to W$.

Any invertible linear transformation $T : V \to W$ is said to be an **isomorphism** from V onto W.

A coordinate mapping is an isomorphism.

Some Facts About Isomorphism

1. Any vector space, V, is isomorphic to itself because the identity transformation $E : V \to V$ is an isomorphism from V onto V.

2. If V is isomorphic to W, then W is isomorphic to V. This is because if $T : V \to W$ is an isomorphism that $T^{-1} : W \to V$ is also an isomorphism. We can say

   "V and W are isomorphic to each other."

3. If V is isomorphic to W and W is isomorphic to X, then V is isomorphic to X.

4. The symbol "$\cong$" is sometimes used, as in $\mathbb{P}_3 \cong R^4$.

Theorem

Suppose that V and W are finite-dimensional vector spaces. Then V and W are isomorphic to each other if and only if $\dim(V) = \dim(W)$. Specifically, if V and W both have dimension k (where $1 \le k < \infty$), then V and W are both isomorphic to R^k.

Remark: This tells us that no matter what sort of objects a finite dimensional subspace S of some vector space contains, we can do stuff using matrices in $R^{\dim(S)}$.

Example

If $M_{3\times 2}$ is isomorphic to R^k, find k.

We need to know how many vectors are in any basis of $M_{3\times 2}$.

$\begin{bmatrix} x & x \\ x & x \\ x & x \end{bmatrix}$ an obvious basis would contain

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\dim(M_{3\times 2}) = 6 \implies k = 6.$$

Example

Determine whether the given pairs are isomorphic or not.

1. $\mathbb{P}_2$ and R^2 no $\dim(\mathbb{P}_2) = 3$

2. $\mathbb{P}_2$ and R^3 yes $\dim(\mathbb{P}_2) = 3 = \dim(R^3)$

3. $\text{Span}\{1, x\}$ and $\text{Span}\{e^{2x}, e^{3x}\}$ 2 dimensional

   yes, both are

4. $M_{3\times 3}$ and R^9 yes $\dim(M_{3\times 3}) = 9$

Chapter 6 Eigenstuff

Consider the linear transformation $T : R^2 \to R^2$ defined by $A\vec{x}$ with
$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$.

1. Evaluate $T(\langle -1, 1\rangle)$ $= \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \langle -1, 1 \rangle = \langle -6, -2 \rangle$

2. Evaluate $T(\langle 1, 3\rangle)$ $= \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \langle 1, 3 \rangle = \langle 2, 6 \rangle$

$T(\langle 1, -2 \rangle) = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \langle 1, -2 \rangle = \langle 7, 1 \rangle$

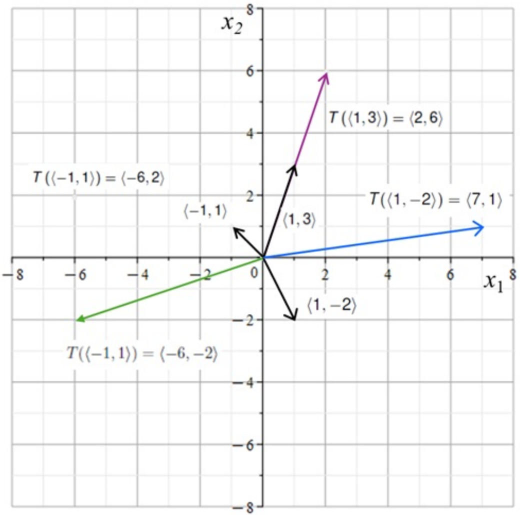

Figure: Plot of standard representations of $\langle -1,1\rangle$,$T(\langle -1,1\rangle)$, $\langle 1,3\rangle$, $T(\langle 1,3\rangle)$, $\langle 1,-2\rangle$, and $T(\langle 1,-2\rangle)$ together.

Example Continued... $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$

Show that there is a special set of nonzero vectors in R^2 with the property $A\vec{x} = 2\vec{x}$.

Let $\vec{x} = \langle x_1, x_2 \rangle$

$A\vec{x} = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \langle x_1, x_2 \rangle = \langle 5x_1 - x_2, 3x_1 + x_2 \rangle$

$2\vec{x} = \langle 2x_1, 2x_2 \rangle$. We want x_1, x_2 such that

$$5x_1 - x_2 = 2x_1 \qquad (5-2)x_1 - x_2 = 0$$
$$3x_1 + x_2 = 2x_2 \quad \Rightarrow \quad 3x_1 + (1-2)x_2 = 0$$

$$\Rightarrow \quad 3x_1 - x_2 = 0$$
$$3x_1 - x_2 = 0$$

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$$

This has coefficient matrix

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} = A - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad x_1 = \frac{1}{3}x_2, \quad x_2 - \text{free}.$$

solutions

$$\vec{x} = x_2 \left\langle \frac{1}{3}, 1 \right\rangle$$

All vectors in $\text{Span}\left\{ \left\langle \frac{1}{3}, 1 \right\rangle \right\}$ will satisfy $A\vec{x} = 2\vec{x}$.

Note $\langle 1, 3 \rangle$ is an example when $x_2 = 3$.

What are Eigen *things*?

We'll focus on linear transformations $T : R^n \to R^n$ with square matrix A such that $T(\vec{x}) = A\vec{x}$, and consider equations of the form

$$A\vec{x} = \lambda\vec{x}$$

with λ a scalar and $\vec{x}$ a nonzero vector. We'll call scalars like λ **eigenvalues** and vectors like $\vec{x}$ **eigenvectors**. We'll also consider things like **eigenspaces** and **eigenbases**.

Questions:

- Does a matrix have such scalars and vectors?
- How would we find them?
- What do the tell us about a matrix?
- What can we do with them?

Determinant

The determinant is a scalar valued function on $M_{n\times n}$. The determinant is related to various properties of a matrix, most notably invertibility.

2×2 Determinant

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The **determinant** of A, denoted $\det(A)$, is the number

$$\det(A) = ad - bc.$$

Example: Evaluate the determinant of each matrix.

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$$

$\det(A) = 5(1) - 3(-1) = 8$ $\qquad$ $\det(B) = 3(-1) - 3(-1) = 0$

Submatrices

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The notation

$$A_{ij}$$

will denote the $(n-1)\times(n-1)$ matrix obtained from A by removing the i^{th} row and the j^{th} column.

For example, if $A = [a_{ij}]$ is a 3×3 matrix, we can form nine different 2×2 matrices A_{ij}.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \rightarrow \quad A_{23} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}.$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \rightarrow \quad A_{31} = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

Minors & Cofactors

Let A be an $n \times n$ matrix, $n \geq 2$. The ijth **minor** of A is the determinant of the $(n-1) \times (n-1)$ matrix A_{ij}. That is, $\det(A_{ij})$ is the ijth minor of A.

Let A be an $n \times n$ matrix, $n \geq 2$.

$$ij\text{th } \textbf{cofactor} \text{ of } A \;=\; (-1)^{i+j}\det(A_{ij}).$$

$n \times n$ Determinant

Let $A = \left[a_{ij}\right]$ be an $n \times n$ matrix. The determinant of A, denoted $\det(A)$ is given by

$$\det(A) = \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det(A_{1j}). \tag{1}$$

The sum in equation (1) is called a **cofactor expansion** across the first row of A.

Cofactor Signs

The factor $(-1)^{i+j}$ gives an alternating sign based on the position of a_{ij} in the matrix.

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Example

$\det(A) = (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{1+2} a_{12} \det(A_{12}) + (-1)^{1+3} a_{13} \det(A_{13})$

Find $\det(A)$ if $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \\ 1 & -1 & 0 \end{bmatrix}$.

$A_{11} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

$\det(A_{11}) = 3(0) - (-1)(2) = 2$

$A_{12} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ $\det(A_{12}) = 0 - 2 = -2$

$A_{13} = \begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix}$ $\det(A_{13}) = 0(-1) - 1(3) = -3$

$\det(A) = (+1)1(2) + (-1)2(-2) + (+1)(-1)(-3)$

$= 2 + 4 + 3 = 9$

Alternative Cofactor Expansions

We can actually take a determint by a cofactor expansion across any row. Fix a number for i (between 1 and n).

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}). \tag{2}$$

Similarly, we can compute $\det(A)$ using a cofactor expansion down any columns. Fix a number j (between 1 and n).

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}). \tag{3}$$

Example

Find $\det(A)$ if $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ by cofactor expansion across the second row and down the third column.

$$\det(A) = (-1)^{2+1} a_{21} \det(A_{21}) + (-1)^{2+2} a_{22} \det(A_{22}) + (-1)^{2+3} a_{23} \det(A_{23})$$

$$= -(0) \det \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} + 3 \det \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} - 2 \det \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= 0 + 3(0+1) - 2(-1-2) = 3 + (-2)(-3) = 3+6 = 9$$

$$\det(A) = 9$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\det(A) = (-1)^{1+3} a_{13} \det(A_{13}) + (-1)^{2+3} a_{23} \det(A_{23}) + (-1)^{3+3} a_{33} \det(A_{33})$$

$$\det(A) = (-1) \det\begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix} - 2 \det\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} + (0) \det\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= -1(0-3) - 2(-1-2) + 0$$

$$= 3 + 6 = 9$$

$$\det(A) = 9$$

Properties of the Determinant

Evaluate $\det(A)$ if $A = \begin{bmatrix} 2 & 0 & -1 & 4 \\ 4 & 0 & 5 & -3 \\ -6 & 0 & 1 & 1 \\ 2 & 0 & 13 & -1 \end{bmatrix}$.

doing cofactor expansion down column 2.

$\det(A) = 0$

Triangular Matrices

A matrix $A = [a_{ij}]$ is called **upper triangular** if $a_{ij} = 0$ for all $i > j$, and it's called **lower triangular** if $a_{ij} = 0$ for all $i < j$. As the names suggest, upper triangular matrices have all their nonzero entries on or above the main diagonal, and lower triangular matrices have all their nonzero entries on or below the main diagonal.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}}_{\text{upper triangular}} \quad \underbrace{\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}}_{\text{lower triangular}}$$

A matrix that is both upper triangular and lower triangular is called a **diagonal** matrix.

$$\underbrace{\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}}_{\text{diagonal matrix}}$$

Since a cofactor expansion is the same across any row or down any column...

Theorem

Let A be an $n \times n$ matrix.

1. If $\vec{0}_n$ is a row vector or a column vector of A, then $\det(A) = 0$.
2. $\det(A^T) = \det(A)$.
3. If A is a triangular matrix (upper, lower or diagonal), the $\det(A)$ is the product of the diagonal entries

$$\det(A) = a_{11}a_{22}\cdots a_{nn}$$

Row Operations

Suppose A is an $n \times n$ matrix.

- If B is obtained from A by performing one row scaling, $kR_i \to R_i$, then $\det(B) = k\det(A)$.
- If B is obtained from A by performing one row swap, $R_i \leftrightarrow R_j$, then $\det(B) = -\det(A)$.
- If B is obtained from A by performing one row replacement, $kR_i + R_j \to R_j$, then $\det(B) = \det(A)$.

Products

If A and B are $n \times n$ matrices, then

$$\det(AB) = \det(A)\det(B)$$

Examples

Find $\det(A)$, $\det(B)$, $\det(AB)$ and $\det(BA)$ where $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

$\det(A) = 5(1) - 3(-1) = 8$

$\det(B) = 2(2) - 0(1) = 4$

$AB = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ 6 & 5 \end{bmatrix}$ $\quad \det(AB) = 10(5) - 6(3)$

$= 32$

$BA = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 13 & -1 \\ 6 & 2 \end{bmatrix}$ $\quad \det(BA) = 13(2) - 6(-1)$

$= 26 + 6$

$= 32$

$\det(A)\det(B) = 8(4) = 32$

$= \det(AB) = \det(BA).$

Example

Suppose A is an invertible $n \times n$ matrix. Show that $\det(A^{-1}) = (\det(A))^{-1}$.

Note $AA^{-1} = I_n$ and $\det(I_n) = 1$

$\det(AA^{-1}) = \det(I_n) = 1$

$\det(A)\det(A^{-1}) = 1$

Bonus result!

This implies $\det(A) \neq 0$ and $\det(A^{-1}) \neq 0$.

$\Rightarrow \det(A) = \dfrac{1}{\det(A^{-1})} = (\det(A^{-1}))^{-1}$

6.2 Eigenvalues & Eigenvectors

Definition

Let A be an $n \times n$ matrix. An **eigenvalue** of A is a scalar λ for which there exists a nonzero vector $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}. \tag{4}$$

For a given eigenvalue λ, a nonzero vector $\vec{x}$ satisfying equation (4) is called an **eigenvector** corresponding to the eigenvalue λ.

Remark 1: Note that eigenvectors are, by definition, **nonzero** vectors.

Remark 2: Eigenvalues are not restricted and can be positive, negative or zero.

Example

Last time, we saw that if we started knowing $\vec{x} = \langle -1, 1 \rangle$ is an eigenvector of $A = \begin{bmatrix} -2 & 2 \\ 7 & 3 \end{bmatrix}$, we can use the equation

$$A\vec{x} = \lambda\vec{x}$$

to find the eigenvalue $\lambda = -4$.

We also found that if we know that $\lambda = 5$ is an eigenvalue, we can use the same equation, $A\vec{x} = \lambda\vec{x}$, to find the eigenvectors $\vec{x} = t\langle 2, 7 \rangle$ with $t \neq 0$.

Questions:

- How do we find them without knowing any λ values or vectors up front?
- Does a matrix A always have eigenvalues and eigenvectors?

No Eigenvalues or Eigenvectors

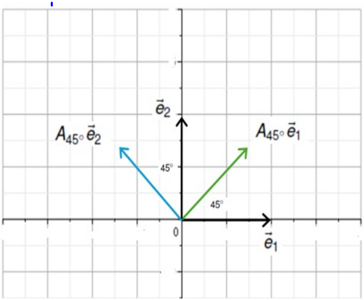

Figure: The matrix $A_{45^\circ} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ rotates each vector 45° counterclockwise.

Question: Could $A_{45^\circ}\vec{x} = \lambda\vec{x}$ for any **nonzero** vector $\vec{x}$ and some number λ?

No, $A_{45^\circ}\vec{x}$ will not be parallel to $\vec{x}$.

The Characteristic Equation

How do we actually find these numbers and vectors? We actually start by finding the eigenvalues. Let's derive a way to find λ such that

$A\vec{x} = \lambda\vec{x}$

Note if B is an $n \times n$ matrix, then

$\det(B) = 0$ if and only if B is not invertible.

Start w/ $A\vec{x} = \lambda\vec{x}$, since $I_n\vec{x} = \vec{x}$

$A\vec{x} = \lambda I_n \vec{x} \Rightarrow A\vec{x} - \lambda I_n \vec{x} = \vec{0}_n$

$\Rightarrow (A - \lambda I_n)\vec{x} = \vec{0}_n$ Non zero $\vec{x}$ will exist

$\iff A - \lambda I_n$ is not invertible $\iff \det(A - \lambda I_n) = 0.$

Definition

Let A be an $n \times n$ matrix. The function

$$P_A(\lambda) = \det(A - \lambda I_n)$$

is called the **characteristic polynomial** of the matrix A. The equation

$$P_A(\lambda) = 0, \quad \text{i.e.,} \quad \det(A - \lambda I_n) = 0$$

is called the **characteristic equation** of the matrix A.

A the name suggests, $P_A(\lambda)$ is always a polynomial in λ. The degree matches the size of the matrix, n, and the leading coefficient is 1 if n is even and -1 if n is odd.

Example

Find the characteristic polynomial of $A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$.

$$A - \lambda I_3 = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-\lambda & 3 & -1 \\ 1 & 2-\lambda & 2 \\ 0 & 0 & -3-\lambda \end{bmatrix}$$

Cofactor expansion across row 3

$$\det(A - \lambda I_3) = (-3-\lambda) \det \begin{bmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{bmatrix}.$$

$$= (-3-\lambda)\left[(4-\lambda)(2-\lambda) - (1)(3)\right]$$

$$= (-3-\lambda)\left[\lambda^2 - 6\lambda + 8 - 3\right]$$

$$= -(3+\lambda)(\lambda^2 - 6\lambda + 5)$$

$$= -(3+\lambda)(\lambda-1)(\lambda-5)$$

$$P_A(\lambda) = -(3+\lambda)(\lambda-1)(\lambda-5)$$

Theorem

Let A be an $n \times n$ matrix, and let $P_A(\lambda)$ be the characteristic polynomial of A. The number λ_0 is an eigenvalue of A if and only if $P_A(\lambda_0) = 0$. That is, λ_0 is an eigenvalue of A if and only if it is a root of the characteristic equation $\det(A - \lambda I_n) = 0$.

Finding the eigenvalues can be challenging if A is a large matrix (high degree polynomial). Once an eigenvalue is known, we can find eigenvectors by solving the homogeneous equation

$$(A - \lambda I_n)\vec{x} = \vec{0}_n$$

using row reduction. This means that the eigenvectors are the nonzero

vectors in $\mathcal{N}(A - \lambda I_n)$.

Example $A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

The characteristic polynomial was

$$P_A(\lambda) = -(3+\lambda)(\lambda-5)(\lambda-1) = -\lambda^3 + 3\lambda^2 + 13\lambda - 15.$$

Find an eigenvector for each eigenvalue.

$P_A(\lambda) = -(3+\lambda)(\lambda-5)(\lambda-1) = 0 \Rightarrow$ $\lambda_1 = -3$, $\lambda_2 = 5$, $\lambda_3 = 1$

For $\lambda_1 = -3$, $A - (-3)I_n = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 7 & 3 & -1 \\ 1 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 7 & 3 & -1 & 0 \\ 1 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & -11/32 & 0 \\ 0 & 1 & 15/32 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = \frac{11}{32} x_3$

$x_2 = \frac{-15}{32} x_3$

x_3 is free

$$A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

Eigenvectors corresponding to $\lambda_1 = -3$ are $\vec{x} = x_3 \left\langle \frac{11}{32}, \frac{-15}{32}, 1 \right\rangle$

An example is $\vec{x} = \langle 11, -15, 32 \rangle$.

For $\lambda_2 = 5$, $A - 5I_3 = \begin{bmatrix} -1 & 3 & -1 \\ 1 & -3 & 2 \\ 0 & 0 & -8 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} -1 & 3 & -1 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 0 & -8 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 3x_2$
x_2 - free
$x_3 = 0$

Eigenvectors are $\vec{x} = x_2 \langle 3, 1, 0 \rangle$.

An example is $\vec{x} = \langle 3, 1, 0 \rangle$.

$$A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

For $\lambda_3 = 1$

$$A - 1I_3 = \begin{bmatrix} 3 & 3 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 3 & -1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2$

x_2 - free

$x_3 = 0$

Eigenvectors are $\vec{x} = x_2 \langle -1, 1, 0 \rangle$

An example is $\vec{x} = \langle -1, 1, 0 \rangle$.

Eigenspaces & Eigenbases

Definition

Let A be an $n \times n$ matrix and λ_0 be an eigenvalue of A. The **eigenspace corresponding to the eigenvalue** λ_0 is the set

$$E_A(\lambda_0) = \{\vec{x} \in R^n \mid A\vec{x} = \lambda_0\vec{x}\} = \mathcal{N}(A - \lambda_0 I_n).$$

An **eigenspace** is a null space, so it's a subspace of R^n. We can find a basis the way we regularly find the basis for a null space.

An **eigenspace** is all of the eigenvectors for a given eigenvalue with the zero vector thrown in to make a subspace.

Example

Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$. Find the characteristic polynomials $P_A(\lambda)$ and $P_B(\lambda)$.

$$P_A(\lambda) = \det(A - \lambda I_3) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{bmatrix}$$

$$= (2-\lambda)(2-\lambda)(4-\lambda) = (2-\lambda)^2(4-\lambda)$$

$$P_B(\lambda) = \det(B - \lambda I_3) = \det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{bmatrix}$$

$$= (2-\lambda)^2(4-\lambda)$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \quad P_A(\lambda) = P_B(\lambda) = (2-\lambda)^2(4-\lambda)$$

$\lambda_1 = 2, \quad \lambda_2 = 4$

Find bases for the eigenspaces $E_A(2)$ and $E_B(2)$.

we need bases for $\mathcal{N}(A - 2I_3)$ and $\mathcal{N}(B - 2I_3)$

$$A - 2I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$(A - 2I_3)\vec{x} = \vec{0} \Rightarrow x_1, x_2$ are free $x_3 = 0$.

$\vec{x} = \langle x_1, x_2, 0 \rangle = x_1 \langle 1, 0, 0 \rangle + x_2 \langle 0, 1, 0 \rangle$

A basis for $E_A(2)$ is $\{ \langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle \}$.

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}, \quad B - 2I_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{rref}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (B - 2I_3)\vec{x} = \vec{0} \implies$$

x_1 - free $\qquad x_2 = x_3 = 0$

$$\vec{x} = \langle x_1, 0, 0 \rangle = x_1 \langle 1, 0, 0 \rangle$$

A basis for $E_B(2)$ is $\{ \langle 1, 0, 0 \rangle \}$.

Two Types of Multiplicities

Geometric Multiplicity

Let A be an $n \times n$ matrix and λ_0 be an eigenvalue of A. The dimension of the eigenspace, $\dim(E_A(\lambda_0))$, corresponding to λ_0 is called the **geometric multiplicity** of λ_0.

To determine the geometric multiplicity, we have to find the dimension of the eigenspace—i.e., how many free variables are there in the equation

$$(A - \lambda_0 I_n)\vec{x} = \vec{0}_n.$$

Algebraic Multiplicity

Let A be an $n \times n$ matrix and λ_0 be an eigenvalue of A. The **algebraic multiplicity** of λ_0 is its multiplicity as the root of the characteristic equation $P_A(\lambda) = 0$. That is, if $(\lambda - \lambda_0)^k$ is a factor of $P_A(\lambda)$ and $(\lambda - \lambda_0)^{k+1}$ is not a factor of $P_A(\lambda)$, then the algebraic multiplicity of λ_0 is k.

If the characteristic polynomial was $(3-\lambda)^4(7-\lambda)^2(-2-\lambda)$, the eigenvalues with their algebraic multiplicities would be

$$\lambda = 3 \text{ algebraic multiplicity } 4,$$
$$\lambda = 7 \text{ algebraic multiplicity } 2,$$
$$\lambda = -2 \text{ algebraic multiplicity } 1.$$

The algebraic multiplicity is always greater than or equal to the geometric multiplicity.

Example

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$P_A(\lambda) = (2-\lambda)^2(4-\lambda) \quad \text{and} \quad P_B(\lambda) = (2-\lambda)^2(4-\lambda).$$

Both have eigenvalue $\lambda = 2$ with **algebraic multiplicity** of two.

$$\underbrace{\{\langle 1,0,0\rangle, \langle 0,1,0\rangle\}}_{\text{basis for } E_A(2)} \qquad \underbrace{\{\langle 1,0,0\rangle\}}_{\text{basis for } E_B(2)}$$

$\lambda = 2$ has geometric multiplicity **two** as an eigenvalue of A and it has a geometric multiplicity **one** as an eigenvalue of B.

If a matrix has enough linearly independent eigenvectors, we may be able to build a basis for R^n out of eigenvectors. So the geometric multiplicity is of interest as is their linear independence.

Linear Independence of Eigenvectors

Suppose A has distinct eigenvalues, λ_1 and λ_2 with corresponding eigenvectors $\vec{x}_1$ and $\vec{x}_2$. Show that $\{\vec{x}_1, \vec{x}_2\}$ is linearly independent.

Consider the equation $c_1\vec{x}_1 + c_2\vec{x}_2 = \vec{0}_n$ *

we need to show that $c_1 = 0$, $c_2 = 0$ is the only solution. we'll create two equations from *.

Multiply * by A.

$$A(c_1\vec{x}_1 + c_2\vec{x}_2) = A\vec{0}_n$$

$$c_1 A\vec{x}_1 + c_2 A\vec{x}_2 = \vec{0}_n$$

Since $A\vec{x}_1 = \lambda_1\vec{x}_1$ and $A\vec{x}_2 = \lambda_2\vec{x}_2$

$$\boxed{c_1 \lambda_1 \vec{X}_1 + c_2 \lambda_2 \vec{X}_2 = \vec{O}_n} \quad (1)$$

Assume $\lambda_1 \neq 0$ (since at least one λ must be non zero.)

Multiply * by λ_1

$$\lambda_1 (c_1 \vec{X}_1 + c_2 \vec{X}_2) = \lambda_1 \vec{O}_n$$

$$\boxed{c_1 \lambda_1 \vec{X}_1 + c_2 \lambda_1 \vec{X}_2 = \vec{O}_n} \quad (2)$$

Subtract equation (2) from equation (1).

$$\vec{O}_n + c_2 \lambda_2 \vec{X}_2 - c_2 \lambda_1 \vec{X}_2 = \vec{O}_n$$

$$c_2 (\lambda_2 - \lambda_1) \vec{X}_2 = \vec{O}_n$$

So $c_2 = 0$ or $\lambda_2 - \lambda_1 = 0$ or $\vec{x}_2 = \vec{0}_n$

$\vec{x}_2$ is an eigenvector so $\vec{x}_2 \neq \vec{0}_n$.

$\lambda_2 - \lambda_1 \neq 0$ since $\lambda_1 \neq \lambda_2$.

Hence $c_2 = 0$.

$*$ becomes $c_1 \vec{x}_1 = \vec{0}_n$

Since $\vec{x}_1$ is an eigenvector, $\vec{x}_1 \neq \vec{0}_n$

and $c_1 = 0$.

Since $c_1 = c_2 = 0$ is the only solution to

$*$, $\{\vec{x}_1, \vec{x}_2\}$ is lin. independent.

Theorem

Let $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ be a set of eigenvectors of an $n \times n$ matrix corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then the set $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is linearly independent.

Note

If A is an $n \times n$ matrix with n distinct eigenvalues, then A has a set of n linearly independent eigenvectors.

Definition

Let A be an $n \times n$ matrix. If A has n linearly independent eigenvectors, $\vec{x}_1, \ldots, \vec{x}_n$ (combined across all eigenvalues), then the set $\mathcal{E}_A = \{\vec{x}_1, \ldots, \vec{x}_n\}$ is a basis for R^n called an **eigenbasis** for A.

Suppose A is $n \times n$

- If A has n distinct eigenvalues, it is guaranteed to have an eigenbasis.
- If A has fewer than n distinct eigenvalues, then
  - it has an eigenbasis if the sum of all geometric multiplicities is n;
  - it doesn't have an eigenbasis if the sum of all geometric multiplicities is smaller than n.

Example

Find an eigenbasis for $A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}$ or show that it is not possible.

Find the eigenvalues.

$$A - \lambda I_2 = \begin{bmatrix} -2-\lambda & 8 \\ 1 & 5-\lambda \end{bmatrix}$$

$$\det(A - \lambda I_2) = (-2-\lambda)(5-\lambda) - (1)(8)$$

$$= \lambda^2 - 3\lambda - 10 - 8$$

$$= \lambda^2 - 3\lambda - 18 = (\lambda - 6)(\lambda + 3)$$

A has two eigenvalues $\lambda_1 = 6$ and $\lambda_2 = -3$.

with two distinct eigenvalues, A will have an eigenbasis.

$$A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}$$

For $\lambda_1 = 6$, $A - 6I_2 = \begin{bmatrix} -8 & 8 \\ 1 & -1 \end{bmatrix}$

$A - 6I_2 \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $(A - 6I_2)\vec{x} = \vec{0}_2$

$x_1 = x_2$, x_2 free

A basis for $E_A(6)$ is $\{\langle 1, 1\rangle\}$.

For $\lambda_2 = -3$, $A - (-3)I_2 = \begin{bmatrix} 1 & 8 \\ 1 & 8 \end{bmatrix}$

$A + 3I_2 \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 8 \\ 0 & 0 \end{bmatrix}$ $(A + 3I_2)\vec{x} = \vec{0}_2$

$x_1 = -8x_2$, x_2 - free

A basis for $E_A(-3)$ is $\{\langle -8, 1\rangle\}$.

A eigenbasis for A is

$$\mathcal{E}_A = \{\langle 1, 1\rangle, \langle -8, 1\rangle\}.$$