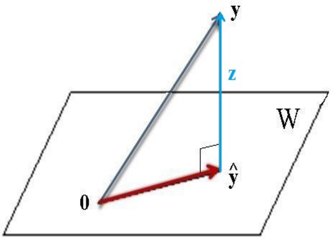
July 20 Math 3260 sec. 51 Summer 2023

Given some point $\mathbf{y}$ in $\mathbb{R}^n$, we considered the problem of finding the point in some subspace W of $\mathbb{R}^n$ that is closest to $\mathbf{y}$, or figuring out how far $\mathbf{y}$ is from W.



$$\mathrm{dist}(\mathbf{y}, \hat{\mathbf{y}}) = \|\mathbf{z}\|$$

Perhaps W is the column space of some matrix, and we want to approximate the solution to a problem $A\mathbf{x} = \mathbf{b}$ that's not actually consistent.

Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$. Each vector $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely as a sum

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^\perp$.

If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is **any orthogonal basis** for W, then

$$\hat{\mathbf{y}} = \sum_{j=1}^{p} \left(\frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \right) \mathbf{u}_j, \quad \text{and} \quad \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}.$$

This holds even if W is a line $\mathrm{Span}\{\mathbf{u}\}$ for one nonzero $\mathbf{u}$ in which case

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u}$$

Orthogonal Decomposition with Orthonormal Basis

If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an **orthonormal** basis of a subspace W of $\mathbb{R}^n$, and $\mathbf{y}$ is any vector in $\mathbb{R}^n$ then

$$\mathrm{proj}_W\, \mathbf{y} = \sum_{j=1}^{p} (\mathbf{y} \cdot \mathbf{u}_j)\, \mathbf{u}_j.$$

And, if U is the matrix $U = [\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_p]$, then the above is equivalent to

$$\mathrm{proj}_W\, \mathbf{y} = UU^T\mathbf{y}.$$

Remark

If W has dimension ≥ 2, these decomposition formulas only work with an orthogonal basis. If you don't have an orthogonal basis, it's not clear how to make use of this.

Section 6.4: Gram-Schmidt Orthogonalization

Big Question:

Given any-old basis for a subspace W of $\mathbb{R}^n$, can we construct an orthogonal basis for that same space?

Example: Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \right\}$. Find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ that spans W.

$\vec{v}_1$ and $\vec{v}_2$ are in W, so

$\vec{v}_1 = a_1 \vec{x}_1 + a_2 \vec{x}_2$ and $\vec{v}_2 = b_1 \vec{x}_1 + b_2 \vec{x}_2$

we need $\vec{v}_1 \cdot \vec{v}_2 = 0$

Let $a_1 = 1$ and $b_2 = 1$ and $a_2 = 0$

so $\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = b_1 \vec{x}_1 + \vec{x}_2$

we need $\vec{v}_1 \cdot \vec{v}_2 = 0$

$$\vec{v}_1 \cdot \vec{v}_2 = \vec{x}_1 \cdot \left(b_1 \vec{x}_1 + \vec{x}_2 \right) = 0$$

$$b_1 \vec{x}_1 \cdot \vec{x}_1 + \vec{x}_1 \cdot \vec{x}_2 = 0$$

$$b_1 \vec{x}_1 \cdot \vec{x}_1 = -\vec{x}_1 \cdot \vec{x}_2$$

$$b_1 = \frac{-\vec{x}_1 \cdot \vec{x}_2}{\vec{x}_1 \cdot \vec{x}_1}$$

Since $\vec{x}_1 = \vec{v}_1$, $b_1 = \dfrac{-\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1}$.

$\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = b_1 \vec{x}_1 + \vec{x}_2$

$$=\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

$\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = \vec{x}_2 - \dfrac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$

$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{x}_1} \vec{x}_2$

$\vec{v}_1 \cdot \vec{x}_2 = -2$

$\vec{v}_1 \cdot \vec{v}_1 = 3$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} - \left(\dfrac{-2}{3}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix}$$

The new, orthogonal basis is

is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix} \right\}$.

Check for othogonality:

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{2}{3} - \frac{1}{3} - \frac{1}{3} = 0 \quad \checkmark$$

Theorem: Gram Schmidt Process

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be any basis for the nonzero subspace W of $\mathbb{R}^n$. Define the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ via

$$
\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \sum_{j=1}^{p-1}\left(\frac{\mathbf{x}_p \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}\right)\mathbf{v}_j.
\end{aligned}
$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W. Moreover, for each $k = 1, \ldots, p$

$$\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \mathrm{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}.$$

Theorem: Gram Schmidt Process

Let $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ be any basis for the nonzero subspace W of $\mathbb{R}^n$. Define the set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ via

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \text{proj}_{\text{Span}\{\mathbf{v}_1\}} \mathbf{x}_2 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \text{proj}_{\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}} \mathbf{x}_3 \\
\mathbf{v}_4 &= \mathbf{x}_4 - \text{proj}_{\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}} \mathbf{x}_4 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \text{proj}_{\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_{p-1}\}} \mathbf{x}_p
\end{aligned}$$

Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthogonal basis for W. If an orthonormal basis is desired, normalize the vectors by setting

$$\mathbf{w}_i = \frac{1}{\|\mathbf{v}_i\|}\mathbf{v}_i, \quad i = 1, \dots, p.$$

The set $\{\mathbf{w}_1, \dots, \mathbf{w}_p\}$ is an **orthonormal** basis for W.

Example

Find an orthonormal (that's *orthonormal* not just orthogonal) basis for

Col A where $A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$.

Note: $A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

The columns of A are a basis for Col A.

Let $\vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{x}_2 \cdot \vec{v}_1 = -6 - 24 - 2 - 4 = -36$$

$$\vec{v}_1 \cdot \vec{v}_1 = 1 + 9 + 1 + 1 = 12$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \left(\frac{-36}{12} \right) \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Note $\vec{v}_1 \cdot \vec{v}_2 = -3 + 3 + 1 - 1 = 0$ ✓

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$\vec{v}_1 \cdot \vec{v}_1 = 12, \quad \vec{v}_2 \cdot \vec{v}_2 = 12$$

$$\frac{\vec{v}_1 \cdot \vec{x}_3}{\vec{v}_1 \cdot \vec{v}_1} = \frac{6}{12} = \frac{1}{2}$$

$$\vec{v}_1 \cdot \vec{x}_3 = -6 + 9 + 6 - 3 = 6$$

$$\vec{v}_2 \cdot \vec{x}_3 = 18 + 3 + 6 + 3 = 30$$

$$\frac{\vec{v}_2 \cdot \vec{x}_3}{\vec{v}_2 \cdot \vec{v}_2} = \frac{30}{12} = \frac{5}{2}$$

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}.$$

$$6 + \frac{1}{2} - \frac{15}{2} = -1$$

$$3 - \frac{3}{2} - \frac{5}{2} = -1$$

$$6 - \frac{1}{2} - \frac{5}{2} = 3$$

$$-3 - \frac{1}{2} + \frac{5}{2} = -1$$

Check:

$$\vec{v}_1 \cdot \vec{v}_3 = 1 - 3 + 3 - 1 = 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = -3 - 1 + 3 + 1 = 0$$

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}$$ is

an orthogonal basis for Col A.

Normalize to get an

orthonormal basis.

$\|\vec{v}_i\| = \sqrt{12}$ for all $i = 1, 2, 3$

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}$$

An orthonormal basis is

$$\left\{ \begin{bmatrix} \frac{-1}{\sqrt{12}} \\ \frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} \frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{-1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{12} \\ -1/\sqrt{12} \\ 3/\sqrt{12} \\ -1/\sqrt{12} \end{bmatrix} \right\}.$$