July 21 Math 3260 sec. 51 Summer 2025

6.2 Eigenvalues & Eigenvectors

We defined eigenvalues and eigenvectors.

Definition

Let A be an $n \times n$ matrix. An **eigenvalue** of A is a scalar λ for which there exists a nonzero vector $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}. \tag{1}$$

For a given eigenvalue λ, a nonzero vector $\vec{x}$ satisfying equation (1) is called an **eigenvector** corresponding to the eigenvalue λ.

We Defined the Characteristic Polynomial/Equation

Definition

Let A be an $n \times n$ matrix. The function

$$P_A(\lambda) = \det(A - \lambda I_n)$$

is called the **characteristic polynomial** of the matrix A. The equation

$$P_A(\lambda) = 0, \quad \text{i.e.,} \quad \det(A - \lambda I_n) = 0$$

is called the **characteristic equation** of the matrix A.

Theorem

Let A be an $n \times n$ matrix, and let $P_A(\lambda)$ be the characteristic polynomial of A. The number λ_0 is an eigenvalue of A if and only if $P_A(\lambda_0) = 0$. That is, λ_0 is an eigenvalue of A if and only if it is a root of the characteristic equation $\det(A - \lambda I_n) = 0$.

Eigenspaces & Eigenbases

Definition

Let A be an $n \times n$ matrix and λ_0 be an eigenvalue of A. The **eigenspace corresponding to the eigenvalue** λ_0 is the set

$$E_A(\lambda_0) = \{\vec{x} \in R^n \mid A\vec{x} = \lambda_0 \vec{x}\} = \mathcal{N}(A - \lambda_0 I_n).$$

An **eigenspace** is a null space, so it's a subspace of R^n. We can find a basis the way we regularly find the basis for a null space.

An **eigenspace** is all of the eigenvectors for a given eigenvalue with the zero vector thrown in to make a subspace.

For $n \times n$ matrix A with eigenvalue λ_0:

Algebraic & Geometric Multiplicities

The **algebraic multiplicity** of λ_0 is its multiplicity as the root of the characteristic equation $P_A(\lambda) = 0$.

The **geometric multiplicity** of λ_0 is the dimension of the eigenspace $E_A(\lambda_0)$.

The algebraic multiplicity of an eigenvalue is greater than or equal to the geometric multiplicity.

To find the multiplicities

- For the algebraic, factor the characteristic polynomial, $P_A(\lambda)$, and
- for the geometric, find a basis for the eigenspace $\mathcal{N}(A - \lambda_0 I_n)$

Theorem

Let $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ be a set of eigenvectors of an $n \times n$ matrix corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then the set $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is linearly independent.

Note

If A is an $n \times n$ matrix with n distinct eigenvalues, then A has a set of n linearly independent eigenvectors.

Remark: For a given matrix, A, we're interested in whether it is possible to build a basis for R^n using only eigenvectors of A.

Definition: Eigenbasis

Let A be an $n \times n$ matrix. If A has n linearly independent eigenvectors, $\vec{x}_1, \ldots, \vec{x}_n$ (combined across all eigenvalues), then the set $\mathcal{E}_A = \{\vec{x}_1, \ldots, \vec{x}_n\}$ is a basis for R^n called an **eigenbasis** for A.

Suppose A is $n \times n$

- If A has n distinct eigenvalues, it is guaranteed to have an eigenbasis.
- If A has fewer than n distinct eigenvalues, then
  - it has an eigenbasis if the sum of all geometric multiplicities is n;
  - it doesn't have an eigenbasis if the sum of all geometric multiplicities is smaller than n.

Example

Find an eigenbasis for $A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}$ or show that it is not possible.

Find the eigenvalues.

$$\det(A - \lambda I_2) = \det \begin{bmatrix} -2-\lambda & 8 \\ 1 & 5-\lambda \end{bmatrix}$$

$$= (-2-\lambda)(5-\lambda) - 8(1)$$

$$= \lambda^2 - 3\lambda - 10 - 8 = \lambda^2 - 3\lambda - 18$$

$$P_A(\lambda) = \lambda^2 - 3\lambda - 18 \quad \text{set } P_A(\lambda) = 0$$

$$(\lambda - 6)(\lambda + 3) = 0 \implies \begin{matrix} \lambda_1 = 6 \\ \lambda_2 = -3 \end{matrix}$$

$$A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}$$

$\lambda_1 = 6$ $\quad A - 6I_2 = \begin{bmatrix} -8 & 8 \\ 1 & -1 \end{bmatrix}$ $\quad (A - 6I_2)\vec{x} = \vec{0}_2$

$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 = x_2$

x_2 - free

$\vec{x} = x_2 \langle 1, 1 \rangle$

A basis for $E_A(6)$ is $\{ \langle 1, 1 \rangle \}$.

$\lambda_2 = -3$, $\quad A - (-3) I_2 = \begin{bmatrix} 1 & 8 \\ 1 & 8 \end{bmatrix}$ $\quad (A + 3I_2)\vec{x} = \vec{0}_2$

$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 8 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 = -8x_2$

x_2 - free

$\vec{x} = x_2 \langle -8, 1 \rangle$

A basis for $E_A(-3)$ is $\{\langle -8, 1 \rangle\}$.

A has an eigenbasis,

$\{\langle 1, 1 \rangle, \langle -8, 1 \rangle\}$.

Example: $A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}$ $\quad \lambda_1 = 6 \quad \lambda_2 = -3$

$\vec{x}_1 = \langle 1, 1 \rangle \quad \vec{x}_2 = \langle -8, 1 \rangle$

1. Create a matrix C having the eigenvectors as its column vectors.
2. Find C^{-1}.
3. Find the product $C^{-1}AC$.

$C = \begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix}$

2. $[C \mid I_2] = \left[\begin{array}{cc|cc} 1 & -8 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$ $\quad -R_1 + R_2 \to R_2$

$\left[\begin{array}{cc|cc} 1 & -8 & 1 & 0 \\ 0 & 9 & -1 & 1 \end{array}\right]$ $\quad \frac{1}{9} R_2 \to R_2$

$$\left[\begin{array}{cc|cc} 1 & -8 & 1 & 0 \\ 0 & 1 & -\frac{1}{9} & \frac{1}{9} \end{array}\right] \quad 8R_2 + R_1 \to R_1$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{9} & \frac{8}{9} \\ 0 & 1 & -\frac{1}{9} & \frac{1}{9} \end{array}\right] \quad C^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 8 \\ -1 & 1 \end{bmatrix}$$

Check: $$C^{-1}C = \frac{1}{9}\begin{bmatrix} 1 & 8 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = I_2 \quad \checkmark$$

$$A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix} \qquad C^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 8 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix}$$

$$C^{-1}AC = \frac{1}{9}\begin{bmatrix} 1 & 8 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix}$$

$\lambda_1 = 6$

$\lambda_2 = -3$

$$= \frac{1}{9}\begin{bmatrix} 1 & 8 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 6 & 24 \\ 6 & -3 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 54 & 0 \\ 0 & -27 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & -3 \end{bmatrix}$$

If $D = \begin{bmatrix} 6 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

D and A are similar matrices.

6.3 Diagonalization

Definition

An $n \times n$ matrix A is said to be **diagonalizable** if it is similar to a diagonal matrix. That is, A is diagonalizable if there exists a diagonal matrix D and an invertible matrix C such that

$$D = C^{-1}AC.$$

The previous example suggests that diagonalizability is related to making a matrix out of eigenvectors. This turns out to be true, but to get an $n \times n$ matrix that is actually invertible, we need n linearly independent vectors. This is where having an eigenbasis comes in.

Facts About Similar Matrices

Theorem

If A and B are similar matrices, the $\det(A) = \det(B)$.

Theorem

If A and B are similar matrices, then A and B have the same eigenvalues, each with the same algebraic and geometric multiplicities.

If A and B are similar, so they share an eigenvalue λ, the eigenvectors corresponding to λ are **generally different**.

$B = C^{-1}AC$

Show that $\det(B) = \det(A)$ and $P_B(\lambda) = P_A(\lambda)$.

$$\det(B) = \det(C^{-1}AC).$$

$$= \det(C^{-1})\det(A)\det(C).$$

$$= \det(C^{-1})\det(C)\det(A)$$

$$= \frac{1}{\det(C)}\det(C)\det(A)$$

$$= 1\det(A) = \det(A).$$

Is $C^{-1}AC = C^{-1}CA$?

NO!

$$P_B(\lambda) = \det(T - \lambda I_n), \quad P_A(\lambda) = \det(A - \lambda I_n)$$

$$\det(B - \lambda I_n) = \det(C^{-1}AC - \lambda I_n)$$

Note $I_n = C^{-1} I_n C$

$$\det(B - \lambda I_n) = \det(C^{-1}AC - \lambda C^{-1} I_n C)$$

$$= \det((C^{-1}A - \lambda C^{-1} I_n) C)$$

$$= \det(C^{-1}(A - \lambda I_n) C)$$

$$= \det(C^{-1}) \det(A - \lambda I_n) \det(C)$$

$$= \det(A - \lambda I_n)$$

$$\Rightarrow \quad P_A(\lambda) = P_B(\lambda).$$

Theorem

Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

Moreover, if A is diagonalizable, then there exists a diagonal matrix D such that $D = C^{-1}AC$ where the columns of the invertible matrix C are the vectors in an eigenbasis, $\mathcal{E}_A$, for the matrix A, and the diagonal entries of the matrix D are the eigenvalues of A.

Remark: If A has n distinct eigenvalues, then it is guaranteed to be diagonalizable. If it has less than n distinct eigenvalues, it may or may not be diagonalizable. A is diagonalizable if the sum of the geometric multiplicities is n.

Example

Let $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. The characteristic polynomial $P_A(\lambda) = (1-\lambda)(2+\lambda)^2$. Determine whether A is diagonalizable.

A has two eigenvalues $\lambda_1 = 1$, $\lambda_2 = -2$.

Find bases for the eigenspaces $E_A(1)$ and $E_A(-2)$.

$\lambda_2 = -2$, $A - (-2)I_3 = \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{rref}}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$x_1 = -x_2$
x_2 - free , $\vec{x} = x_2 \langle -1, 1, 0 \rangle$.
$x_3 = 0$

The geometric multiplicity of $\lambda_2 = -2$ is one. A is not diagonalizable.

Example

Diagonalize the matrix $A = \begin{bmatrix} -4 & 3 \\ -6 & 5 \end{bmatrix}$ if possible.

$P_A(\lambda) = (\lambda - 2)(\lambda + 1) \Longrightarrow$ $\lambda_1 = 2$ $\quad \vec{x} = x_2 \left(\frac{1}{2}, 1\right)$

$\lambda_2 = -1$ $\quad \vec{x} = x_2 \langle 1, 1 \rangle$

A basis for $E_A(2)$ is $\{\langle 1, 2 \rangle\}$.

A basis for $E_A(-1)$ is $\{\langle 1, 1 \rangle\}$.

A is diagonalizable.

$C = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, $\quad D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

$$C^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$C^{-1}AC = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{pmatrix} -4 & 3 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \checkmark$$

Evaluate A^{10} if $A = \begin{bmatrix} -4 & 3 \\ -6 & 5 \end{bmatrix}$.

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = C^{-1} A C = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A = C D C^{-1} \qquad D^{10} = \begin{bmatrix} 2^{10} & 0 \\ 0 & (-1)^{10} \end{bmatrix} = \begin{bmatrix} 1024 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{10} = C D^{10} C^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1024 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1024 & 1024 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1022 & 1023 \\ -2046 & 2047 \end{bmatrix}$$