July 6 Math 3260 sec. 51 Summer 2023

4.1–4.3 Recap

- We defined vector spaces & subspaces.
- For an $m \times n$ matrix A, we defined some associated subspaces of $\mathbb{R}^n$ (the null and row spaces) or $\mathbb{R}^m$ (the column space).
- We generalized the concept of a **linear transformation** so that the domain and codomains can be any vector spaces (not only $\mathbb{R}^{\text{something}}$), and defined the kernel and range.

4.1–4.3 Recap

- We defined linear dependence/independence for a set of vectors.
- We defined a **basis** (a linearly independent spanning set) for a vector space or subspace.
- We found that an rref can be used to construct bases for each of the null, row and column spaces of a matrix.

Basis

Definition:

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a **basis** of H provided

(i) $\mathcal{B}$ is linearly independent, and

(ii) $H = \text{Span}(\mathcal{B})$.

Remark: A **basis** is a set of building blocks that can be used to build a vector space using just the two operations, vector addition and scalar multiplication.

Remark: The linear independence requirement means that every vector in the basis gives some necessary information about the vector space.

Bases for $\text{Col}(A)$, $\text{Row}(A)$, and $\text{Nul}(A)$

Given a matrix A, find the rref. Then

- The pivot columns of the original matrix A give a basis for $\text{Col}(A)$.
- The nonzero rows of rref(A) give a basis for $\text{Row}(A)$.
- Use the rref to solve $A\mathbf{x} = \mathbf{0}$ to identify a basis for $\text{Nul}(A)$.

Section 4.4: Coordinate Systems

Theorem:

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis for a vector space V. Then for each vector $\mathbf{x}$ in V, there is a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n.$$

- **Remark:** It's clear that each vector can be written as a linear combination because a basis is a spanning set.

- **Remark:** This is saying that it can only be done in one way—that is, there is only one set of numbers $c_1, \ldots, c_n$.

Uniqueness of Coefficients

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis for a vector space V and let $\mathbf{x}$ be a vector in V. If

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots c_n\mathbf{b}_n \quad \text{and} \\ \mathbf{x} &= a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots a_n\mathbf{b}_n, \end{aligned}$$

show that $a_1 = c_1, a_2 = c_2, \ldots, a_n = c_n$.

We can use a homogeneous equation

because $\vec{x} - \vec{x} = \vec{0}$. Subtract

$$\vec{0} = (c_1 - a_1)\vec{b}_1 + (c_2 - a_2)\vec{b}_2 + \cdots + (c_n - a_n)\vec{b}_n$$

We have a homogeneous vector

equation in $\vec{b}_1, \dots, \vec{b}_n$

Since $\{\vec{b}_1, \dots, \vec{b}_n\}$ is linearly independent the coefficients must all be zero.

$$c_1 - a_1 = 0 \Rightarrow a_1 = c_1$$

$$c_2 - a_2 = 0 \Rightarrow a_2 = c_2$$

$$\vdots$$

$$c_n - a_n = 0 \Rightarrow a_n = c_n$$

Consequence of Linear Independence

$$\text{Let} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It is true that $\mathbb{R}^2 = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Consider $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Note that we can write $\mathbf{x}$ in two different ways

$$\begin{aligned} \mathbf{x} &= 2\mathbf{v}_1 + 3\mathbf{v}_2 + 0\mathbf{v}_3 \quad \text{and} \\ \mathbf{x} &= 1\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3. \end{aligned}$$

Why doesn't this contradict our theorem?

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is lin. dependent

Definition: Coordinate Vectors

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an **ordered** basis of the vector space V. For each $\mathbf{x}$ in V we define the **coordinate vector of x relative to the basis** $\mathcal{B}$ to be the unique vector $(c_1, \ldots, c_n)$ in $\mathbb{R}^n$ whose entries are the weights $\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n$.

We'll use the notation $[\mathbf{x}]_{\mathcal{B}}$; that is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$.

Big Idea!

The vector $\mathbf{x}$ can be any sort of vector (from any sort of vector space), but

$$\boxed{[\mathbf{x}]_{\mathcal{B}} \quad \text{is a vector in} \quad \mathbb{R}^n}$$

Example

Consider the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ (in that order) for $\mathbb{P}_3$.
Determine $[\mathbf{p}]_{\mathcal{B}}$ for

(a) $\mathbf{p}(t) = 3 - 4t^2 + 6t^3$

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \text{ if } \vec{p}(t) = c_1(1) + c_2(t) + c_3(t^2) + c_4(t^3)$$

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ -4 \\ 6 \end{bmatrix}$$

$\mathcal{B} = \{1, t, t^2, t^3\}$

Determine $[\mathbf{p}]_{\mathcal{B}}$ for

(b) $\mathbf{p}(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

An Alternative Basis for $\mathbb{R}^2$

Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find $[\mathbf{x}]_{\mathcal{B}}$ for *

$\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ where $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

as a matrix equation

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$\leftarrow P_{\mathcal{B}} [\vec{x}]_{\mathcal{B}} = \vec{x}$

*If we just write **x** with no brackets around it, the implication is that it's written with respect to the elementary basis $\mathcal{E}=\{\mathbf{e}_1, \mathbf{e}_2\}$.

Let's use matrix inversion

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Change of Coordinates Matrix

Note in this example that

$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

where $P_{\mathcal{B}}$ is the matrix having the basis vectors from $\mathcal{B}$ as its columns.

$$P_{\mathcal{B}} = [\mathbf{b}_1 \ \mathbf{b}_2]$$

Definition

Given an ordered basis $\mathcal{B}$ in $\mathbb{R}^n$, the matrix

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n].$$

is called the **change of coordinates matrix** for the basis $\mathcal{B}$ (or from the basis $\mathcal{B}$ to the standard basis).

Change of Coordinates in $\mathbb{R}^n$

Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. Then the change of coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the linear transformation defined by

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$$

where the matrix

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n].$$

Remark: By the Invertible Matrix Theorem, we know that a change of coordinates mapping is a **one to one** transformation of $\mathbb{R}^n$ **onto** $\mathbb{R}^n$.

Example

$\vec{x} = P_{\mathcal{B}} [\vec{x}]_{\mathcal{B}}$

$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \vec{x}$

For $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$, we have

$$P_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad P_{\mathcal{B}}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

(a) Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \vec{x} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\vec{x} = P_{\mathcal{B}} [\vec{x}]_{\mathcal{B}}$

$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \vec{x}$

(b) Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \vec{x} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(c) Find $\mathbf{x}$ if $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\vec{x} = P_{\mathcal{B}} [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

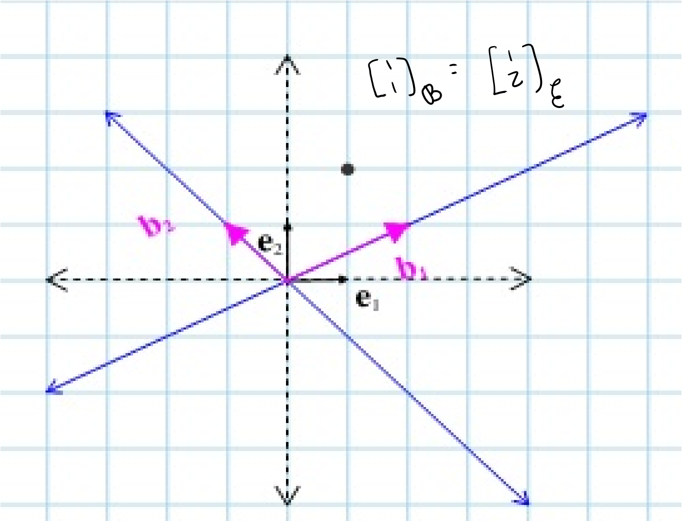

Figure: $\mathbb{R}^2$ shown using elementary basis $\{(1,0),(0,1)\}$ and with the alternative basis $\{(2,1),(-1,1)\}$.

We can make new graph paper using this basis.

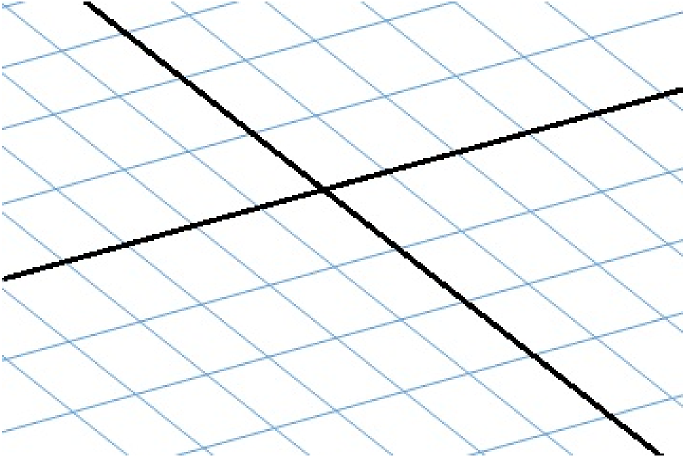

Figure: Graph paper constructed using the basis $\{(2,1),(-1,1)\}$.

Theorem: Coordinate Mapping

Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a **one to one** mapping of V **onto** $\mathbb{R}^n$.

Remark:

- For a vector space V, if there exists a coordinate mapping from $V \to \mathbb{R}^n$, we say that V is **isomorphic** to $\mathbb{R}^n$.
- Properties of subsets of V, such as linear dependence, can be discerned from the coordinate vectors in $\mathbb{R}^n$.

$\mathbb{P}_3$ is **Isomorphic** to $\mathbb{R}^4$

We saw that using the ordered basis $\mathcal{B} = \{1, t, t^2, t^3\}$ that any vector

$$\mathbf{p}(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$$

in $\mathbb{P}_3$ has coordinate vector

$$[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

in $\mathbb{R}^4$.

Example

Use coordinate vectors to determine if the set $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ is linearly dependent or independent in $\mathbb{P}_2$.

$$\mathbf{p}(t) = 1 - 2t^2, \quad \mathbf{q}(t) = 3t + t^2, \quad \mathbf{r}(t) = 1 + t$$

Let's use the elementary basis in $\mathbb{P}_2$

$\mathcal{B} = \{1, t, t^2\}$ in that order.

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad [\vec{q}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \quad [\vec{r}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Let $A = \begin{bmatrix} [\vec{p}]_{\mathcal{B}} & [\vec{q}]_{\mathcal{B}} & [\vec{r}]_{\mathcal{B}} \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

Let's compute $\det(A)$

$$\det(A) = 1 \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 0 & 3 \\ -2 & 1 \end{vmatrix}$$

$$= 1(0-1) + 1(0+6) = 5 \neq 0$$

So the columns of A are linearly independent in $\mathbb{R}^3$.

Hence $\{\vec{p}, \vec{q}, \vec{r}\}$ is linearly independent in $\mathbb{P}_2$.

Section 4.5: Dimension of a Vector Space

Theorem:

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set of vectors in V containing *more than n vectors* is linearly dependent.

This extends our result in $\mathbb{R}^n$ that said that a set with more vectors than entries in each vector had to be linearly dependent.

For example,

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix} \right\}$$

is linearly dependent because there are 4 vectors from $\mathbb{R}^3$.

Example

Recall that a basis for $\mathbb{P}_3$ is $\{1, t, t^2, t^3\}$.

Is the set below linearly dependent or linearly independent?

$$\{1+t, 2t-3t^3, 1+t+t^2, 1+t+t^2+t^3, 2-t+2t^3\}$$

This set contains 5 vectors.

A basis for $\mathbb{P}_3$ has 4 vectors.

This set is linearly dependent.

All Bases are the same Size

Our theorem gives the immediate corollary:

Corollary:

If vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then every basis of V consist of exactly n vectors.

Remark: This makes sense. If one basis had more vectors than another basis, it couldn't be linearly independent.

Dimension

Consider a vector space V.

Definition:

If V is spanned by a finite set, then V is called **finite dimensional**. In this case, the dimension of V

$$\dim V \;=\; \text{the number of vectors in any basis of } V.$$

The dimension of the vector space $\{\mathbf{0}\}$ containing only the zero vector is defined to be zero—i.e.

$$\dim\{\mathbf{0}\} = 0.$$

If V is not spanned by a finite set[a], then V is said to be **infinite dimensional**.

[a] $C^0(\mathbb{R})$ is an example of an infinite dimensional vector space.

Examples

(a) Determine $\dim(\mathbb{R}^n)$.

The elementary basis has n vectors in it.

$$\dim(\mathbb{R}^n) = n$$

(b) Determine $\dim \operatorname{Col}(A)$ where $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$.

There are two pivot columns, so a basis has two vectors in it.

$$\dim(\operatorname{Col} A) = 2$$

Some Geometry in $\mathbb{R}^3$

We can describe all of the subspaces of $\mathbb{R}^3$ geometrically. The subspace(s) of dimension

(a) zero: is just the origin (one point), $(0, 0, 0)$.

(b) one: are lines through the origin. $\text{Span}\{\mathbf{u}\}$ where $\mathbf{u}$ is not the zero vector.

(c) two: are planes that contain the origin and two other, noncolinear points. $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ with $\{\mathbf{u}, \mathbf{v}\}$ linearly independent.

(d) three: is all of $\mathbb{R}^3$.

Subspaces and Dimension

Theorem:

Let H be a subspace of a finite dimensional vector space V. Then H is finite dimensional and

$$\dim H \leq \dim V.$$

Moreover, any linearly independent subset of H can be expanded if needed to form a basis for H.

Remark: We said before that we can take a spanning set and remove extra vectors to get a basis. This follow up statement says if we start with a linearly independent set, we can add to it as needed to get a basis.

Subspaces and Dimension

Theorem:

Let V be a vector space with $\dim V = p$ where $p \geq 1$. Any linearly independent set in V containing exactly p vectors is a basis for V. Similarly, any spanning set consisting of exactly p vectors in V is necessarily a basis for V.

Remark: this connects two properties **spanning** and **linear independence**. If $\dim V = p$ and a set contains p vectors then

- linear independence $\Longrightarrow$ spanning
- spanning $\Longrightarrow$ linear independence

Again, this is **IF** the number of vectors matches the dimension of the vector space.

Column and Null Spaces

Theorem:

Let A be an $m \times n$ matrix. Then

$$\dim \operatorname{Nul} A = \text{the number of free variables in the equation } A\mathbf{x} = \mathbf{0},$$

and

$$\dim \operatorname{Col} A = \text{the number of pivot positions in } A.$$

Example

A matrix A is show along with its rref. Find the dimensions of the null space and column space of A.

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & 1 & -7 & -1 \\ 3 & 0 & 6 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\dim(\mathrm{Col}\,A)$ = # pivot columns

$\dim(\mathrm{Nul}\,A)$ = # free variables in $A\vec{x} = \vec{0}$

$\dim(\mathrm{Col}\,A) = 3$, $\dim(\mathrm{Nul}\,A) = 1$

Example

A matrix A along with its rref is shown.

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for Row A and state $\dim \text{Row}\, A$.

$$\text{Row}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -5 \end{bmatrix} \right\}$$

this is a basis

$$\dim(\text{Row}\, A) = 3$$

Example continued ...

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find a basis for $\mathrm{Col}\,A$ and state $\dim \mathrm{Col}\,A$.

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

this is a basis.

$$\dim(\mathrm{Col}\,A) = 3$$

Example continued ...

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) Find a basis for Nul A and state dim Nul A.

From the rref

$x_1 = -x_3 - x_5$

$x_2 = 2x_3 - 3x_5$

$x_4 = 5x_5$

x_3, x_5 - are free

$$\vec{x} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$ is a basis

$$\dim(\mathrm{Nul}\, A) = 2$$